SEQUENTIAL TESTS FOR 2X2 CONTINGENCY TABLES

William Q. Meeker, Jr.

Union College

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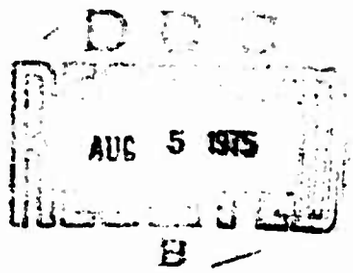

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L.A. Aroian, Project Director

UNION COLLEGE AND UNIVERSITY
Institute of Administration and Management
Schenectady, New York 12308

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This report deals with sequential tests of problems which can be formulated in terms of a 2x2 contingency table. All of the important cases (marginal probabilities known and unknown and marginal populations "observable" and "not observable") are treated. Theory for finding the sequential test regions is developed and the exact values of the important test properties are found using Aroian's direct method of sequential analysis. The tests are compared with fixed size tests and a method of estimation is presented. Numerical examples and computer programs are included.

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INTRODUCTION

This report presents theory and methods for treating sequentially certain problems which can be formulated in terms of 2x2 contingency tables. The report is organized as follows. Chapter 1 contains some preliminary material, including a discussion of the different models which arise with the treatment of 2x2 contingency tables. Chapter 2 treats some general topics related to sequential analysis which are common to all of the models considered here. Chapters 3, 4 and 5 show how to develop sequential tests and evaluate exactly their properties, for three important models of the 2x2 contingency tables. Numerical examples are provided and the tests compared with other similar tests, both fixed size and sequential (when available). Chapter 6 presents a method which can be used to estimate the parameters of a 2x2 contingency table at the termination of a sequential hypothesis test. Chapter 7 summarizes the results, discusses some possible areas for further research and ends with some concluding remarks. Computer programs used to perform the necessary computations are given in the Appendix.

CHAPTER 1

DISCUSSION OF 2x2 TABLES AND REVIEW OF THE LITERATURE

1.0 INTRODUCTION

This chapter introduces 2x2 contingency tables and treats some of the common methods of analysis which have been used for them. In general, 2x2 tables are used to test independence of a bivariate Bernoulli process. The first section discusses, in general, the tests of independence to be considered here. The different types of 2x2 contingency tables can be divided into two broad groups, tables for which the marginal probability functions are known and tables for which marginal probability functions are unknown. These cases are discussed in Sections 1.2 and 1.3 respectively. The fixed size test procedures for these cases are also reviewed. Section 1.4 surveys the different types of problems which can be formulated in terms of a 2x2 table.

Some approximate methods of treating contingency tables (e.g., the χ^2 test) are only appropriate when the sample size is sufficiently large to meet certain conditions. For small samples, some exact methods (i.e., methods which are not based on any asymptotic approximations) have been proposed. It is these exact methods for small samples which are treated sequentially here. The exact methods are, in theory, equally applicable to large samples; however, the necessary computation becomes laborious, if not prohibitive, with presently available computing machinery.

1.1 TESTS OF INDEPENDENCE

This section introduces some of the preliminaries necessary for the treatment given here to sequential tests of 2x2 contingency tables. As explained in detail below, one is interested in testing for independence or for some degree of dependence between the rows and columns of a 2x2 contingency table. Depending on the underlying probability model of the situation being considered, the degree of dependence can be expressed in terms of a single parameter, say θ. There is one particular value of θ, say θ_0, for which the hypothesis of independence is true. There is positive dependence in the table if $\theta<\theta_0$ and negative dependence if $\theta>\theta_0$. The probability models and the particular value of θ to be used for each are described in the following sections.

In a two decision test, the hypothesis might be expressed, for example, as

$$\begin{aligned} H_0&: \theta = \theta_0 \\ H_1&: \theta = \theta_1 \neq \theta_0 \end{aligned} \tag{1.1}$$

H_0 is usually known as the null hypothesis and H_1 is the alternative hypothesis and may be either simple or composite. When testing this hypothesis, there are two types of errors with which one must be concerned. These are shown in Figure 1.1.

Decision Based on Test Results

True State of Nature		H_0	H_1
	H_0	No Error	α Error
	H_1	β Error	No Error

Figure 1.1 Error Probabilities for a Two Decision Test

The first is called a Type I or α error and is made when there is a decision to reject H_0 when it is true; the probability of committing such an error is usually denoted by α. A Type II or β error occurs when the null hypothesis is accepted when in fact some specified alternate hypothesis is true. The probability of such an error is usually denoted by β. The following notation, however, is used here. Let α and β denote the desired probabilities of the Type I and Type II errors respectively and let α' and β' denote the actual error probabilities of the sequential tests.

When a three decision test procedure* is being used, one of the three hypotheses must be selected. These hypotheses can be specified as

$$\begin{aligned} H_1&: \theta = \theta_1 < \theta_0 \\ H_0&: \theta = \theta_0 \\ H_2&: \theta = \theta_2 > \theta_0 \end{aligned} \tag{1.2}$$

*The three decision test is a generalization of the standard two-sided test; that is, separate α and β errors can be specified for each alternate hypothesis (see Goss (1974b)).

In this case, there are four types of errors which can be made; α_1 is the probability of accepting H_1 when H_0 is true and β_1 is the probability of accepting H_0 or H_2 when H_1 is true; α_2 is the probability of accepting H_2 when H_0 is true, and β_2 is the probability of accepting H_1 or H_0 when H_2 is true. These error probabilities are shown in Figure 1.2.

		Decision Based on Test Results		
		H_1	H_0	H_2
True State of Nature	H_1	No Error	β_1 Error	
	H_0	α_1 Error	No Error	α_2 Error
	H_2	β_2 Error		No Error

Figure 1.2 Error Probabilities for a Three Decision Test

The following sections of this chapter will treat the individual cases which arise with 2x2 contingency tables. The underlying probability models are discussed and fixed size procedures are examined. In the succeeding chapters, sequential methods for testing these hypotheses are treated.

1.2 CONTINGENCY TABLES WITH KNOWN MARGINAL PROBABILITIES

The underlying probability model of a 2x2 contingency table is a bivariate Bernoulli process. This is illustrated in Figure 1.3.

		Dark Eyes D	Light Eyes $\bar{D}$	
Dark Hair	E	p_{11}	p_{12}	$p_{1.}$
Light Hair	$\bar{E}$	p_{21}	p_{22}	$p_{2.}$
		$p_{.1}$	$p_{.2}$	1

Figure 1.3 Probabilities in a 2x2 Table

The observations from this model are assumed to be identically and independently distributed. Such a situation arises when one samples from an infinite population (or from a finite population with replacement) and the presence or absence of two attributes is observed at each trial.

If, for example, the event D represents dark eyes and the event E represents dark hair observed on a person selected at random with replacement from a specified population, p_{11}, p_{12}, p_{21} and p_{22} in Figure 1.3 are the joint probabilities of observing the respective combination of attributes. This model is more conveniently represented as in Figure 1.4 which expresses the

	D	$\bar{D}$	
E	p_{11}	$p_{1.}-p_{11}$	$p_{1.}$
$\bar{E}$	$p_{.1}-p_{11}$	$1-p_{.1}-p_{.1}+p_{11}$	$1-p_{1.}$
	$p_{.1}$	$1-p_{.1}$	1

Figure 1.4 Probabilities in a 2x2 Table

model in terms of only three parameters. This notation will be used below.

The test to be performed in this model is of independence between the two characteristics being observed. The null hypothesis of independence can be stated, for example, as

$$p_{11} = p_{1.}p_{.1} \tag{1.3}$$

or

$$t = \frac{p_{11}(1-p_{1.}-p_{.1}+p_{11})}{(p_{.1}-p_{11})(p_{1.}-p_{11})} = 1 \tag{1.4}$$

implying, for the above example, that dark eyes do not tend to occur more often with the characteristic dark hair than with light hair. The statements in (1.3) and (1.4) can be shown to be equivalent.

In this section the marginal probabilities (i.e., $p_{1.}$ and $p_{.1}$) are assumed known. Such a case might occur in the example given above if the characteristics of hair and eye color had been studied independently, but no information is available on the frequency with which they tend to occur together. The underlying distribution can also be expressed as a multinomial distribution with four cells. If the observed data from a sample of size n is represented as in Figure 1.5,

	D	$\bar{D}$	
E	x	$n_{1.} - x$	$n_{1.}$
$\bar{E}$	$n_{.1}-x$	$n-n_{1.}-n_{.1}+x$	$n-n_{1.}$
	$n_{.1}$	$n-n_{.1}$	n

Figure 1.5 Observed Contingency Table

the probability of observing this data can be expressed as

$$P(x,n_{1.},n_{.1};p_{11},p_{1.},p_{.1}) = \tag{1.5}$$

$$\frac{n!p_{11}^{x}(p_{1.}-p_{11})^{n_{1.}-x}(p_{.1}-p_{11})^{n_{.1}-x}(1-p_{1.}-p_{.1}+p_{11})^{n-n_{.1}-n_{1.}+x}}{x!(n_{1.}-x)!(n_{.1}-x)!(n-n_{1.}-n_{.1}+x)!}$$

Because $p_{1.}$ and $p_{.1}$ are assumed known, the hypotheses to be tested are specified as

$$H_0: \ p_{11}=p_{1.}p_{.1} \tag{1.6}$$

$$\text{versus } H_a: \ p_{11}\neq p_{1.}p_{.1}$$

This hypothesis is discussed in detail in Chapter 3 where it is shown that the triplet $(n_{1.},n_{.1},x)$ is a minimal sufficient statistic for the state of nature (p_{11}).

An exact fixed size procedure for small samples can be constructed to test (1.6) by ordering the multinomial probabilities for all of the possible occurrences under the null hypothesis and partitioning off a critical region consisting of those points with the smallest probabilities which favor H_a and which sum to the desired significance level. The power of the test can be found by finding the probability of observing a point in the critical region under specified alternatives to the null hypothesis.

For large samples, the computation necessary for the above tests becomes laborious. The χ^2 distribution provides an easy-to-use approximation to the null distribution of the test. The χ^2 test is constructed in the usual manner (for the approximation of a multinomial distribution) except that the proper number of degrees

of freedom is three because the parameters $p_{1.}$ and $p_{.1}$ are known. Guttman et al. (1971) give an example of the use of the χ^2 approximation for this case; it is also treated by Rao (1952). In Chapter 3, exact sequential tests for such hypotheses are developed.

There are two special cases of 2x2 contingency tables with known marginal probabilities. The first arises when both marginal totals are random variables and only one of the marginal probability distributions is known. Not much treatment seems to have been given to this case in the past. The χ^2 approximation with two degrees of freedom is appropriate for large samples. The other special case arises when one of the marginal distributions is "observable." "Observable" in this case means that the distribution from the margin can be controlled by the experimenter in some way and is not a random variable except in its relations to the sample size in a sequential test. This means that a sequential (or fixed size) test can be constructed such that a desired proportion of units can be taken from each category of the "observable" margin at each stage of the test. Lehmann (1959) points out that tests which take equal numbers from each category are asymptotically most powerful.

The case where one margin is "observable" and the other is random with an unknown probability distribution is treated in the next section. The case where one margin is "observable" and the other is random with a known probability distribution reduces to a simple binomial distribution if one samples exclusively from

one of the characteristics of the "observable" margin. This test can be shown to be asymptotically most powerful (Lehmann, 1959) and can be treated sequentially by using a simple binomial procedure. (See Ghosh (1970), p.282). The case where both margins are "observable" is mentioned briefly in the next section.

1.3 CONTINGENCY TABLES WITH UNKNOWN MARGINAL PROBABILITIES

The treatment of 2x2 tables with unknown marginal probabilities, as described in this section, has been a classical problem in the field of mathematical statistics. It is particularly interesting because of the controversies which have arisen concerning their proper treatment. A brief history of the results obtained with this well-known model is given here. The model considered in this section is the same bivariate Bernoulli process discussed in the last section, except that here both of the marginal probability distributions are assumed to be unknown. The hypothesis of independence being tested, however, is the same. The unknown marginal probabilities $p_{1.}$ and $p_{.1}$ are so-called "nuisance parameters," causing the method of testing with small samples to be quite different. This subject is treated in detail in Chapters 4 and 5.

Karl Pearson (1900) was apparently the first to treat the problem when he suggested the χ^2 distribution as an approximation to the test of independence. This is still the accepted approach when the expected number in each cell is sufficiently large. There was, for a time, some controversy as to the proper number of degrees of freedom to be used for the test. This was settled

by Fisher (1922) and Yule (1922) who show that when the marginal probabilities are unknown, the proper number of degrees of freedom is one.

The use of the χ^2 distribution is an approximation to the true multinomial distribution which assumes the count in each cell of the table to be normally distributed. Because of this, it is necessary that the expected values of the entries in each cell of the table be of sufficient size to justify this assumption. In most cases an expected number of 5 in each cell is considered sufficient for the use of the χ^2 approximation, although this is still a matter of some controversy. A continuity correction for the approximation can also be used. Recent treatment of this subject is given, for example, by Lancaster (1969) and Fleiss (1973).

Fisher (1935) and Yates (1934) concurrently presented a test for 2x2 tables which is exact for small samples. The test is based on the concept of ancillary statistics as defined by Fisher (1935). Briefly, the test is constructed to be conditional on the observed margins. In this case, the distribution of the observations in the table under the null hypothesis of independence reduces to the much simpler hypergeometric distribution. This also produces a much smaller reference set from which to choose the critical region. This test is treated more completely in Sections 4.2 and 5.1.

The Fisher-Yates test (also known as Fisher's exact test) led to a great deal of controversy among some of the most well-known mathematical statisticians, including E.B. Wilson (Wilson, 1941), G.A. Barnard (Barnard, 1945, 1947a and 1947b) and

E.S. Pearson (Pearson, 1947). Their basic disagreement was with Fisher's reference set. Pearson and Barnard believed that the test of significance should be based on all of the possible occurrences from a given sample size. Fisher insisted on limiting the reference set to only those different possible outcomes, given the observed marginal totals. Fisher's argument, based on the concept of ancillary statistics, as an answer to this criticism, is given in Section 4.2; it is now generally agreed that Fisher's method is the one which should properly be used for the above model.

Barnard (1947a) surveys the different types of 2x2 tables with unknown marginal probabilities. He divides the tables into three groups, depending on whether the margin totals are random variables or fixed constants. He terms these "double dichotomy," "2x2 comparative trial" and "2x2 independence trial," for the cases where neither, one, and both margins are fixed (i.e., "observable") respectively. A brief discussion of these models follows.

If both margins are random variables, one is interested in the degree of dependence between the rows and columns. If one of the margins is "observable" as explained in Section 1.2, that margin's totals can be controlled by the experimenter. This is Barnard's "comparative trial" and can be used, for example, to test homogeneity of the two populations with respect to some attribute. Although it is not necessary to do so, if the test is is conducted such that an equal number of observations are taken

from each category of the fixed margin, the asymptotic power of the test for a given significance level can be shown to be a maximum (Lehmann, 1959). An example of such a test would be selecting n/2 people with dark hair and n/2 people with light hair. The proportions of dark-eyed people in each category are then compared.

If a sample of fixed size is selected from each category of one margin, there are two parameters in the model; namely, for the present example, the proportions of dark-eyed people with dark hair and with light hair. The probability model is illustrated in Figure 1.6 where

$$p_1 = p_{11}/p_{1.}$$
$$p_2 = (p_{.1} - p_{11})/(1-p_{1.}) \qquad (1.7)$$

		Dark Eyes D	Light Eyes $\bar{D}$
Dark Hair	E	p_1	$1-p_1$
Light Hair	$\bar{E}$	p_2	$1-p_2$

Figure 1.6 2x2 Table for Testing $p_1=p_2$

This is the common test for the equality of two unknown binomial proportions where the null hypotheses to be tested can be expressed as

$$p_1 = p_2$$

$$\text{or} \quad \frac{p_1(1-p_2)}{p_2(1-p_1)} = 1 \tag{1.8}$$

For fixed size tests with large samples, the normal distribution approximation can be used to test the hypotheses in (1.8). The Fisher's exact test (Fisher, 1935) can be used for small samples to treat this situation. The model can also be formulated in a logistic form. This is done in Chapter 5.

Barnard (1945, and 1947a) gives a test of homogeneity which he claims is "more powerful than Fisher's." In this test, Barnard considers the larger reference set of outcomes mentioned above. The test's introduction was followed by some discussion (Fisher (1945), Barnard (1945, 1947a, 1949)) which led to the general consensus that Fisher's test is the one which should properly be used. Some further treatment of this subject is given in Section 4.2 and Chapter 5, where sequential tests for these cases are presented.

The other case delineated by Barnard is the independence trial, where both of the margin totals are fixed. This is situation illustrated by Fisher's famous tea-tasting experiment where a lady is to decide whether the milk or the tea was put into the cup first. In this test the lady is informed as to how many of the cups are in each category, and it is assumed that her answers will correspond in number. This is again a test concerning the independence of the marginal characteristics. Fisher's exact test

is also used in this case. Because such "fixed" margin models do not often arise in sequential analysis, they are not treated here.

1.4 OTHER PROBLEMS FORMULATED IN TERMS OF 2x2 CONTINGENCY TABLES

This section will survey some of the statistical problems which have been formulated in terms of 2x2 tables. All of these cases have been treated in the literature for fixed size tests. Some of them can be solved sequentially with the methods given here. Others will have to be treated in a somewhat different manner. Some discussion of these possible extensions is contained in Chapter 7.

The three most commonly used models for 2x2 tables are the "double-dichotomous," the "comparative trial" and the "independence trial," as named by Barnard and discussed in Section 1.3. These are models with unknown marginal probabilities (for the random margins) and have 0, 1, and 2 fixed margins respectively. The "double dichotomous" model is used for testing the independence of two Bernoulli processes. The use of such tests is common, for example, in both medical and psychological research. The "comparative trial" is used to test the equality of two unknown binomial proportions, or to test for independence when one of the populations in the "double dichotomous" model is "observable" as explained in Section 1.3. Such tests might be used, for example, to test whether a new drug is significantly more effective than a placebo or another standard. The "independence trial" is

a test of independence between two fixed marginal totals.

The first two cases can often be treated more conveniently with a sequential test. This is especially true if the data are obtained, or if the test is conducted, sequentially. The sequential tests for these cases are developed in Chapters 4 and 5 respectively. The third case has limited applicability within the area of sequential analysis.

It is interesting to note that the fixed size test of the null hypothesis for all of these cases is the same. For small samples, Fisher's exact test (see Chapter 4) can be used, and for large enough samples, the χ^2 distribution with one degree of freedom is appropriate. Two other applications of the "double dichotomous" model are non-parametric tests of location and for dispersion. These tests are treated, for example, by Gibbons (1971) and Owen (1962).

If either or both of the marginal distributions are known, different fixed size procedures are required, as explained in Section 1.1. The sequential procedure to be used when both marginal probability distributions are known is developed in Chapter 3.

In addition to the above, other problems have been formulated in terms of 2x2 tables or combinations of 2x2 tables. Dr. John Gart has been a leader in this field of application. Some of the problems which he has formulated in terms of 2x2 tables include tests for comparing matched proportions in crossover designs (Gart, 1969) comparison of several proportions adjusted

for an auxiliary variable or covariate, and test of incidence rates when the underlying distribution can be assumed to be Poisson (Gart, 1974).

CHAPTER 2

SEQUENTIAL ANALYSIS AND THE DIRECT METHOD

2.0 INTRODUCTION

This chapter introduces and reviews some of the important topics and considerations relating to sequential analysis which are used in the sequential tests for 2x2 contingency tables treated in Chapters 3, 4 and 5. The first section discusses the use of sequential analysis when testing composite hypotheses and the basic importance of the operating characteristic (OC) function. Section 2.2 introduces the direct method of sequential analysis which is used later to find the exact properties of the sequential tests. The next section treats different methods of developing sequential tests for three decision test procedures. The last section explains the truncation of sequential tests to eliminate the possibility of very large sample sizes.

2.1 SEQUENTIAL ANALYSIS AND COMPOSITE HYPOTHESES

This section will consider sequential tests of composite hypotheses. It will be shown here that the Wald (1947) sequential probability ratio test (SPRT), used in the following chapters and based on pairs of simple hypotheses, can be used to obtain satisfactory sequential tests for composite hypotheses. The discussion below pertains to two decision tests, although the ideas also apply to $k>2$ decision tests.

When finding a fixed size sample test to choose between one of two specified hypotheses, one must specify both the sample size n* and critical value c* to give the desired error probabbilities. When this special case is generalized to a sequential procedure where stopping rules are selected for each trial, the problem of selection of the proper test becomes much more complicated because there are many more possible tests to choose from. To find a sequential test, one must partition the sample space at each trial into three regions: one for acceptance of H_0, one for rejection of H_0 and one for continuation of the sequential test.

It is well known that Wald SPRT gives optimum regions for testing a simple hypothesis against a simple alternative under certain conditions (Wald and Wolfowitz, 1948). Such hypotheses are stated, using the binomial parameter p for an example, as

$$H_0: \ p=p_0 \qquad \text{versus} \qquad H_1: \ p=p_1 \tag{2.1}$$

as shown in Figure 2.1. The hypotheses are represented as points if they are simple, as in this case, and as line segments if they are composite. For our purpose, we define simple and composite hypotheses to be hypotheses specifying exactly one point (in the parameter space), and more than one point, respectively. Statistical tests between two alternative simple hypotheses imply that the experimenter believes that there are only two possible values for the true state of nature. Such situations do not often occur in practice.

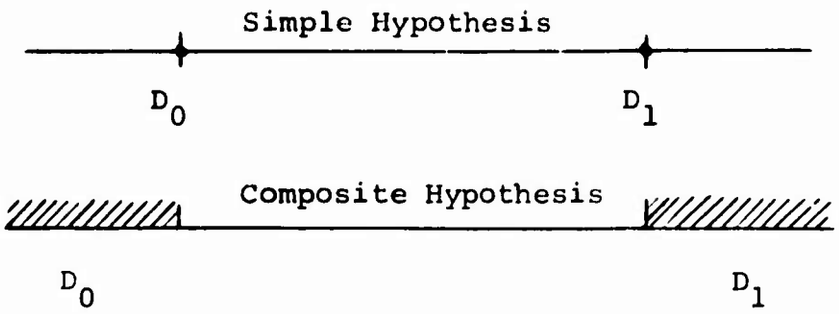

Figure 2.1 Simple and Composite Hypotheses

In most cases the hypotheses to be tested are composite and are expressed in a form similar to

$$H_0: p=p_0 \quad \text{versus} \quad H_1: p\neq p_0 \tag{2.2}$$

$$\text{or} \quad H_0: p<p_0 \quad \text{versus} \quad H_1: p>p_1\geq p_0 \tag{2.3}$$

When using a statistical test, the important distinction between the simple hypotheses in (2.1) and the composite hypotheses of (2.2) and (2.3) is that in the latter one is interested in all of the points of the OC function over a specified range of the parameter values given by the hypothesized states of nature.

The hypotheses shown in (2.2) do not contain any <u>specific</u> alternative and are the type generally specified in so-called fixed size sample "tests of significance." Users of such tests generally use a specified significance level (α error) and sample size, but do not mention a specific alternative hypotheses and therefore often do not consider the "power" of their tests. The rationale for such a test is that there is a strong prior

belief in (or preference for) the null hypothesis, and that it is not to be rejected unless there is strong evidence (i.e., at the $1-\alpha$ confidence level) that it is not true.

By examining the Type II error (which is one minus the power of the test at a specific alternative), one can determine if the significance level of the test has been set too low (or too high) for a given sample size or if the sample size is too large (or too small) for the required sensitivity against alternatives to the null hypothesis. Either of these consequences could be costly. It does no harm for even the "significance tester" to investigate to which his alternatives his test will be sensitive. From this it is seen that it is important to examine the power of a statistical test.

In this light, the pair of hypotheses in (2.3) is considered. Here a range of values has been specified for H_1, the alternative hypothesis, as well as for H_0, the null hypothesis (see Figure 2.1). The values in between p_0 and p_1 constitute an "indifference zone." For the situation where one must make a decision either for H_0 or for H_1, and there are positive costs (tangible or not) for both types of errors, this is a more practical way of specifying the hypothesis to be tested.

This again brings out the subtle difference between a "test of significance" and other composite tests of hypotheses. A test of significance might be valid, for example, for a test used in proving some law of nature, for which it is nearly impossible to specify all of the possible alternatives. In contrast, when testing the ability of a new drug to cure a disease, the situation is different.

If the proportion of successful cures of a drug is to be compared with that of a control or a placebo, the hypotheses to be tested will usually be stated as

$$H_0: \ p_1 = p_2 \qquad (2.4)$$
$$\text{versus } H_1: \ p_1 < p_2$$

where p_1 and p_2 are the probabilities of a successful cure for the control and the drug being tested, respectively (both probabilities being unknown). In this case, there are true costs (although they are probably intangible) for both types of errors; that is, for accepting the new drug as "significantly better"* when it is not and for rejecting it when it is "significantly better." Because both of these errors are important, it is imperative that the experimenter examine the power of his statistical test so that the errors can be balanced if necessary. These same ideas are important in the development of sequential tests of composite hypotheses.

When developing sequential tests, it is usually necessary to specify some specific alternatives to the null hypothesis, so that the proper stopping rules can be formulated to control both types of errors and so that the test properties of the sequential test may be assessed. If one wishes to test a composite hypothesis such as (2.3), one must find a sequential test procedure which has

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*Here we mean a difference of practical significance, rather than simply a difference of statistical significance.

a satisfactory OC function over a specified range of parameter values. This is usually done with respect to some additional criterion concerning the cost of sampling.

Although the Wald procedure provides optimal tests under certain conditions, there remains the problem of finding optimum sequential tests for the composite hypotheses considered here. Wald (1947) discusses this problem at some length. He comes to the conclusion that the test of the simple hypothesis in (2.1) can be used to approximate a test of a composite hypothesis such as (2.3) without much loss of efficiency. This is the method most commonly used to find regions for a sequential test of a composite hypothesis. In Chapters 3, 4 and 5, sequential test regions are found by specifying simple hypotheses.

One should examine the possible consequences of using such an approximation; that is, carefully examine the OC function of the test. If the resulting OC function is not close to the desired OC function, the test region can be modified so that it is. This is briefly discussed in Section 3.3.

2.2 THE DIRECT METHOD OF SEQUENTIAL ANALYSIS

The direct method of sequential analysis, given by Aroian (1968), describes a general method whereby the exact properties of a given sequential test region may be obtained. Since Aroian's 1968 article, the method has been used in a variety of applications, including tests for the mean of a normal distribution with the standard deviation known (Aroian and Robison, 1969) and unknown (Schmee, 1974), two sided tests of the normal distribution with the standard deviation known (Goss, 1974b)

sequential rank tests (Elfring and Schultz, 1973b), tests of the binomial distribution (Corneliussen and Ladd, 1970 and 1971), and tests of a normal distribution with mean known and unknown (Aroian, Gorge, Goss and Robison, 1975).

Before using a sequential test procedure, one should know or have available reasonable approximations to the actual test properties. The most important test properties are the true α and β error probabilities (denoted α' and β' here) and the expected or average sample number (ASN). A typical ASN function for a state of nature which can be expressed in one dimension (e.g., the binomial parameter p) is shown in Figure 2.2. Also of interest is the operating characteristic (OC) function which gives the probability of accepting the null hypothesis as a function of the state of nature. A typical OC function for a one-dimensional state of nature is shown in Figure 2.3. If the state of nature must be defined in two dimensions, these functions can be represented as contours or by single graphs with one parameter being held constant. If more than two dimensions are necessary to describe the state of nature, it will be best to show the test properties in tables. The true α and β error probabilities for a two decision procedure are obtained directly from the OC function as

$$\begin{aligned} \alpha' &= 1-OC(p_0) \\ \beta' &= OC(p_1) \end{aligned} \qquad (2.5)$$

where p_0 and p_1 are the parameters specified by the null and alternate hypotheses respectively.

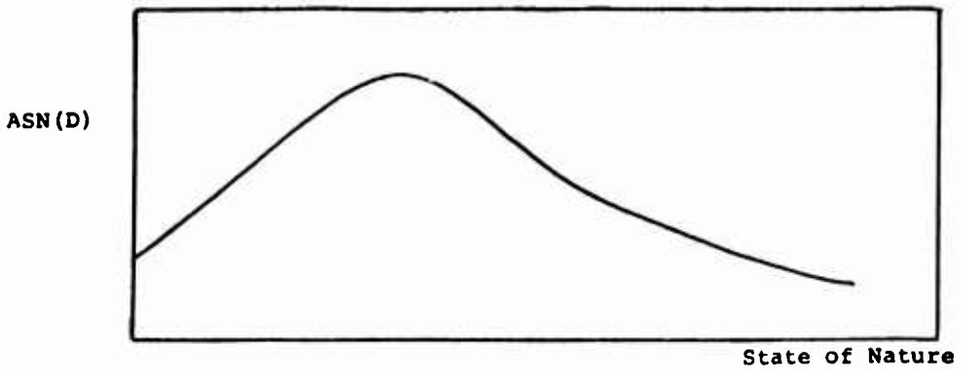

Figure 2.2 Typical ASN Function

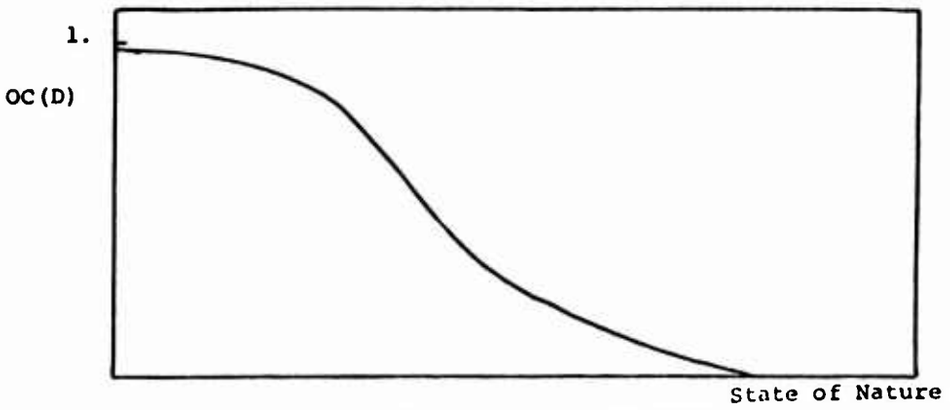

Figure 2.3 Typical OC Function

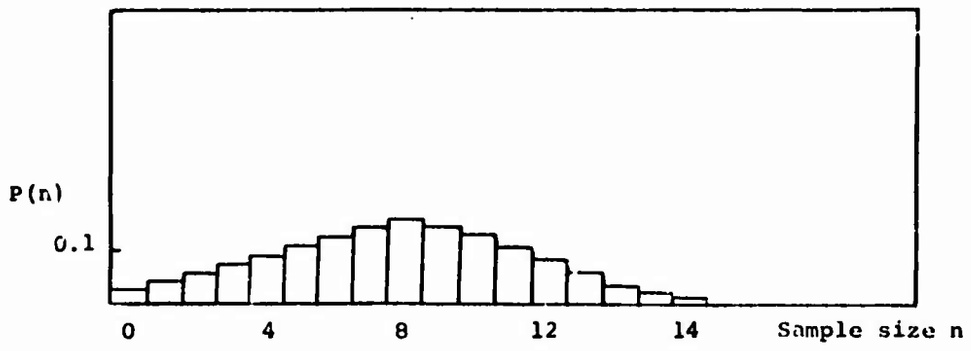

Figure 2.4 Typical Distribution of the DSN

Another interesting test characteristic, which is often neglected, is the distribution of the decisive sample number (DSN); that is, the probability mass function of the sample size necessary to reach a decision. This distribution is a function of the true state of nature. From this distribution, one can obtain the ASN, the variance of the sample number (VSN) or other moments. The direct method is also used to find the distribution of the DSN. A typical probability mass function for the DSN is shown in Figure 2.4.

In general, the direct method is carried out as follows. Once the sequential test region (i.e., the sequential test rules) has been specified, one chooses a state of nature, which allows the computation of the probability of accepting each possible hypothesis at the first trial. The remaining probability, that is, the probability of being in the continuation region, is spread out among all the possible values of the sample statistic which are included in the continuation region. At the second trial, another sample is taken. It is again necessary to find the probability of accepting each hypothesis and the distribution of the probability of remaining in the continuation region. Using convolutions, one may continue this process at each succeeding trial or until the probability of remining in the continuation region is so small as to be insignificant. This entire procedure is then repeated using different values for the true state of nature, each giving a point on the OC function and a distribution of the DSN. This procedure is used in Chapters 3, 4 and 5 to find the exact properties for sequential tests of 2x2 contingency tables.

2.3 METHODS FOR THREE DECISION SEQUENTIAL TEST PROCEDURES

In this section, the procedures for developing three decision sequential tests are reviewed. Three decision tests are often necessary in practice. An example of such a test would be the comparison of two drugs where one might be interested in testing the proportion of successful cures in a controlled test. The hypotheses to be tested might be expressed as

$$\begin{aligned} & H_1: \ p_1 > p_2 \\ \text{versus } & H_0: \ p_1 = p_2 \\ \text{versus } & H_2: \ p_1 < p_2 \end{aligned} \tag{2.6}$$

where p_1 and p_2 represent the proportion of successful cures for drug 1 and 2 respectively. One might also use such a test to distinguish among lots of items which are of superior quality (for which some incentive bonus might be given), standard quality and substandard quality. The hypotheses for this case might be specified as

$$\begin{aligned} & H_1: \ D=D_1 > D_0 \\ \text{versus } & H_0: \ D=D_0 \\ \text{versus } & H_2: \ D=D_2 < D_0 \end{aligned} \tag{2.7}$$

where D represents the number (or proportion) of defectives in the lot.

The following is a brief sketch of the different approaches to three decision tests which have been treated in the literature.

Ghosh (1970) and Goss (1974b) give excellent and somewhat more comprehensive treatment of this subject. The discussion here is general in that it pertains to no specific distribution. No attempt has been made to cover the many applications of these tests. For this, the reader is referred to Wetherill (1966).

Wald (1947), in his book, gives a method of formulating a two-sided test by using weight functions. Barnard (1947c), in his review of Wald's book, mentions an alternate method which simply tests the null hypotheses separately against the two alternatives. This is done by using two SPRTs at one time. The resulting test regions are shown geometrically in Figure 2.5. Sobel and Wald (1949), in their paper, treat the three decision test in detail. They use a test similar to that suggested by Barnard. The difference is that each SPRT is treated independently of the other. This would mean, for example, that when line AB is crossed by the path shown in Figure 2.5, we no longer allow acceptance of H_1 and concern ourselves only with the results of SPRT2. Thus, H_0 is accepted when line AC is crossed at point p, before a shaded region is even reached. Sobel and Wald hasten to point out that such a test, which depends not only on the total sample results, but also on the sample path (order of the observations), cannot be an optimal one. However, the test was used in their case because the independence of the two tests enabled the authors to derive approximations for some of the properties of this three decision test. The Sobel-Wald tests and their approximate properties are treated in detail by Ghosh (1971). Here, we

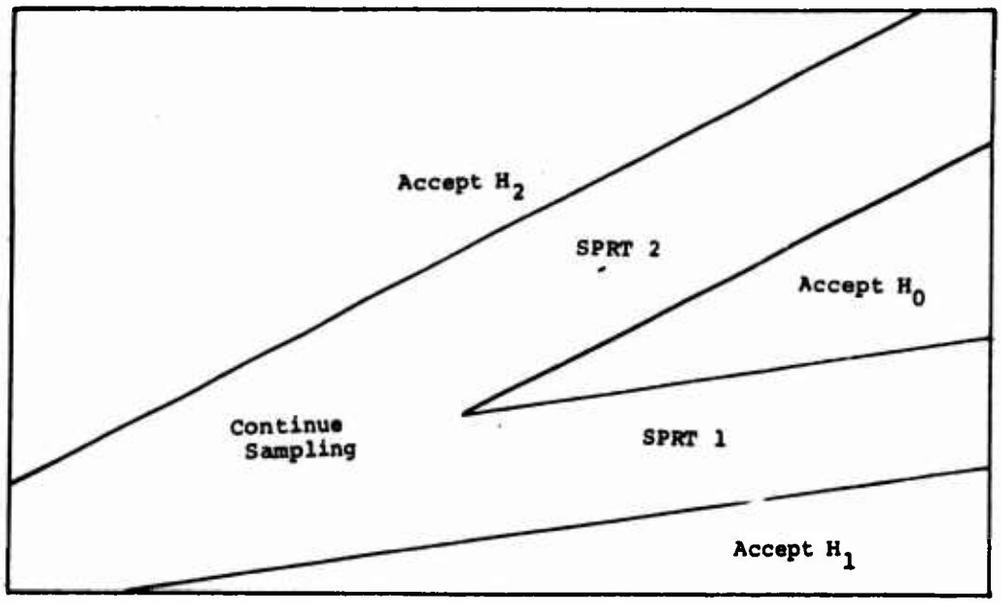

Figure 2.5 A Three Decision Sequential Test Region

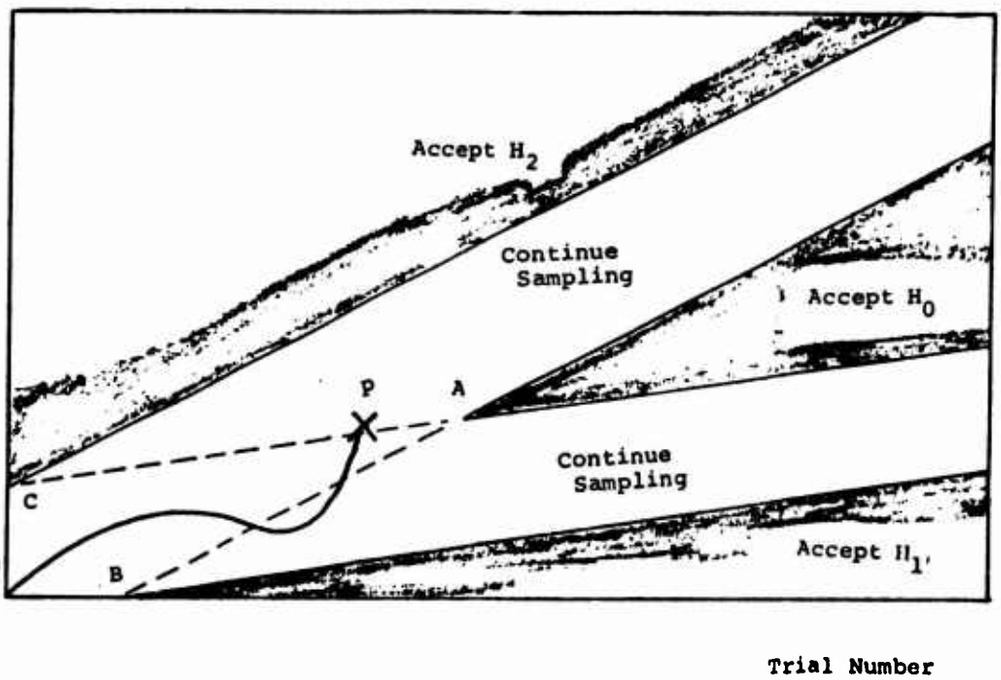

Figure 2.6 Illustration of the Independence of the Sobel-Wald Sequential Test Region

use the direct method of sequential analysis which can be used to find the exact properties of any specified sequential test region.

Goss (1974b), when treating three decision sequential tests of the mean of a normal distribution, compared the Sobel-Wald test with the Barnard test. He used the direct method to obtain exact test results for such tests. From his results, (as one would expect intuitively), it is seen that the test with independently run SPRTs has a smaller expected sample size, but slightly larger error probabilities. The differences, however, are quite small. For this reason and because it has somewhat more intuitive appeal, the approach suggested by Sobel and Wald is used here, although a decision to accept a hypothesis is allowed if and only if one enters a shaded region in Figure 2.5.

Another approach to the three decision test is given by Armitage (1950). In this paper, Armitage suggests using three SPRTs simultaneously. The three SPRTs are constructed to distinguish between H_1 and H_0, H_2 and H_0 and between H_1 and H_2. This is shown graphically in Figure 2.6.

To devise the three decision sequential tests used here, a modified version of the Sobel-Wald procedure (Sobel and Wald, 1949) is used. Following their treatment, two SPRTs are used simultaneously. One SPRT, say SPRT1, is used to distinguish between H_0 and H_1. The other SPRT, say SPRT2, is used to distinguish between H_0 and H_2. The procedure for developing and evaluating the test properties of the three decision test procedures is treated in detail for the special cases of the 2x2 contingency tables in Chapters 3, 4 and 5.

2.4 TRUNCATION OF THE SEQUENTIAL TEST REGIONS

One disadvantage of using sequential test procedures is that because the sample size is a random variable, it is sometimes possible for the sample size to be significantly larger (although with small probability) than the sample size necessary for a fixed size sample test. This section presents methods for truncating sequential tests at some trial, say n_0. This will result in a closed sequential test whose test properties, with respect to the ASN function, will be much improved. The price paid for this improvement is usually quite small.

When one wishes to truncate a sequential test at some trial, say n_0, one must specify which one of the hypotheses is to be chosen for each possible value of the test statistic (which may be multidimensional) at trial n_0. Some general rules of thumb for doing this are given in Section 3.3. Further modification of the region can be made on a trial and error basis, using the exact probabilities (obtained by using the direct method of sequential analysis) of reaching each of the decision points in the sample space as a guide. Such careful modification, though tedious, could be used to obtain a sequential test with test properties closely approaching those which are desired.

Often when truncation procedures are put forward, the truncation point suggested is from 1.5 to 3 times n_0 (see, for example, Wald (1947)). This is probably because in the past, very little was known about the exact properties of such truncated tests. When using the direct method, however, this presents no

problem because the direct method is general and can be used to evaluate any specified test region. The sequential tests presented here are usually truncated at the sample size required for a similar fixed size test (n*) and are compared with such fixed size tests.

When a sequential test is truncated at some trial, say n_0, the true α and β error probabilities will increase by some, usually small, amount (when compared to the untruncated test). If the size of this increase cannot be tolerated, the error probabilities can be reduced in one of two ways. First, the test can be truncated at some trial $n_0 > n^*$ (i.e., at some trial greater than the comparable fixed size test). This, however, will allow the sample size to increase (usually with small probability) above n*. It will also tend to a general increase in the ASN function. The other method is to modify the test region by including more points in the continuation region for trials $n < n_0$. This will enable one to approach the α and β error probabilities of the fixed size test with n_0 trials by increasing the ASN function (which will approach a constant function equal to n_0).

All of the sequential tests for 2x2 contingency tables presented here have been truncated. Some further discussion of the particular methods used to truncate these tests is contained in Section 3.3.

CHAPTER 3

SEQUENTIAL TESTS WHEN THE MARGINAL PROBABILITIES ARE KNOWN

3.0 INTRODUCTION

This chapter treats sequential methods for 2x2 contingency tables when the marginal probabilities are known. Section 3.1 discusses such contingency tables and gives their underlying probability model. Section 3.2 describes the hypothesis being tested. Sections 3.3 to 3.6 present the development and evaluation of the sequential tests for both two and three decision test procedures. Section 3.5 also compares the sequential tests developed here with a comparable fixed size test.

3.1 2x2 CONTINGENCY TABLES AND THE MULTINOMIAL DISTRIBUTION

The underlying probability model for a 2x2 contingency table is depicted in Figure 3.1.

p_{11}	$p_{1.}-p_{11}$	$p_{1.}$
$p_{.1}-p_{11}$	$1-p_{1.}-p_{.1}+p_{11}$	$1-p_{1.}$
$p_{.1}$	$1-p_{.1}$	1

Figure 3.1 Probability model for a 2x2 contingency table

As indicated in Section 1.1, this can be considered a bivariate binomial distribution. The two marginal distributions are independent if and only if $p_{11}=p_{1.}p_{.1}$. One is usually interested in testing the hypothesis of independence, although tests for any degree of association can be easily constructed. A full discussion of these hypotheses is given in the next section.

The probability model in Figure 3.1 can also be expressed as a multinomial distribution. The probability of observing the sample shown in Figure 3.2

	D	$\overline{D}$	
E	x	$n_{1.}-x$	$n_{1.}$
$\overline{E}$	$n_{.1}-x$	$n-n_{.1}-n_{1.}+x$	$n-n_{1.}$
	$n_{.1}$	$n-n_{.1}$	n

Figure 3.2 Sample from a 2x2 contingency table

is then

$$p_F(x,n_{1.},n_{.1};n,p_{1.},p_{.1},p_{11})= \tag{3.1}$$

$$\frac{n!p_{11}^{x}(p_{1.}-p_{11})^{n_{1.}-x}(p_{.1}-p_{11})^{n_{.1}-x}(1-p_{1.}-p_{.1}+p_{11})^{n-n_{1.}-n_{.1}+x}}{x!(n_{1.}-x)!(n_{.1}-x)!(n-n_{.1}-n_{1.}+x)!}$$

Because the marginal probabilities $p_{1.}$ and $p_{.1}$ are known, the state of nature is completely specified by p_{11}. That is, there are no nuisance parameters to deal with, as is the case when one

or both of the marginal probability functions are unknown. The triplet $(x,n_{1.},n_{.1})$ is a minimal sufficient statistic for p_{11}. This can be shown as follows.

In order to show sufficiency one must show that the ratio

$$\frac{P_F(x,n_{1.},n_{.1};n,p_{11},p_{1.},p_{.1})}{P_F(y,m_{1.},m_{.1};n,p_{11},p_{1.},p_{.1})} \tag{3.2}$$

is independent of the state of nature (see Lindgren (1968), p.256). The probability mass function P_F is as defined in (3.1).

Equation (3.2) is independent of the state of nature if and only if

$$\begin{aligned} n_{1.}&=m_{1.} \\ n_{.1}&=m_{.1} \\ \text{and} \quad x&=y \end{aligned} \tag{3.3}$$

The vector $(x,n_{1.},n_{.1})$ therefore is the minimal sufficient statistic for the true state of nature, p_{11}. Sequential tests based on this statistic are presented in subsequent sections.

As mentioned in Section 1.1, a special case arises if one of the marginal distributions is "observable" and one category of that margin can be sampled from exclusively. Because one knows the marginal probability function of the other margin, the problem reduces to a simple binomial distribution which can be used to test association between the two marginal characteristics. This greatly simplifies the problem.

If, for example, $n_{1.}$ can be chosen to be the same as n (the total sample size), the distribution of x is

$$P(x,n_{1.},p') = \binom{n_{1.}}{x} (p')^{x}(1-p')^{n_{1.}-x} \qquad (3.4)$$

and x is a sufficient statistic for p_{11}. The hypothesis to be tested is

$$\begin{aligned} H_0: &\quad p'=p'_0=p_{.1} \\ \text{versus } H_1: &\quad p'=p'_1 \neq p_{.1} \end{aligned} \qquad (3.5)$$

where p' is the conditional probability of obtaining an observation in cell 1 of Figure 3.3, given the observation is in either cell 1 or 2.

	D	$\bar{D}$
E	1	2
$\bar{E}$	3	4

Figure 3.3 2x2 Table Cell Numbers

This hypothesis can be treated sequentially by using a simple binomial test (Wald, 1947).

3.2 THE HYPOTHESIS BEING TESTED

This section discusses the hypothesis being tested in a 2x2 contingency table when marginal probabilities are known. As mentioned in the last section, one is interested in testing independence of two binomial characteristics. The hypothesis can be expressed as

$$H_0: \; p_{11}=p_{1.}p_{.1}$$
$$\text{versus } H_1: \; p_{11}\neq p_{1.}p_{.1} \tag{3.6}$$

As indicated in Section 3.1, p_{11} alone exactly specifies the state of nature in this case. Two other equivalent ways of specifying this null hypothesis are

$$\lambda = \frac{p_{11}}{p_{1.}p_{.1}} = 1$$
$$\text{or} \quad t = \frac{p_{11}(1-p_{1.}-p_{.1}+p_{11})}{(p_{1.}-p_{11})(p_{.1}-p_{11})} = 1 \tag{3.7}$$

The first is the ratio between the two values which are hypothesized as being equal; the second is commonly known as the cross product ratio and is treated in detail in Section 4.2.

Thus there are three methods of specifying the alternate hypothesis to be tested. The ranges of variation of the parameters mentioned above are

$$\text{MAX } (0, p_{1.}+p_{1.}-1) \leq p_{11} \leq \text{MIN } (p_{1.}, p_{.1})$$
$$\text{MAX } \left\{0, \frac{p_{1.}+p_{.1}-1}{p_{1.}p_{.1}}\right\} \leq \lambda \leq \text{MIN } \left\{\frac{1}{p_{1.}}, \frac{1}{p_{.1}}\right\} \tag{3.8}$$
$$0 \leq t < \infty$$

For the purposes of testing the case with known marginal probabilities considered here, specifying the alternate hypothesis directly in terms of p_{11} is most convenient.

A three decision test of independence for the above model can be specified as:

$$\begin{aligned} & H_1: \quad p_{11}=p_1<p_0 \\ \text{versus } & H_0: \quad p_{11}=p_0=p_{1.}p_{.1} \\ \text{versus } & H_2: \quad p_{11}=p_2>p_0 \end{aligned} \tag{3.9}$$

Sequential tests for these hypotheses are treated in Sections 3.4 and 3.6.

3.3 THEORY FOR SEQUENTIAL TESTS WITH TWO DECISIONS

In this section, sequential tests for the two decision hypotheses discussed in the last section are developed. It is assumed that the marginal distributions of the bivariate Bernoulli process are known and that items are sequentially selected at random from a population which follows this distribution. The hypothesis to be tested is:

$$\begin{aligned} & H_0: \quad p_{11}=p_0 \\ \text{versus } & H_1: \quad p_{11}=p_1>p_0 \end{aligned} \tag{3.10}$$

The sequential test for distinguishing between these two simple hypotheses is developed as follows. Following Wald (1947), the sequential test is carried out by calculating the likelihood ratio at trial n, with a sample outcome $(x,n_{1.},n_{.1})$ (see Figure 3.2).

$$Ln_1/Ln_0 = \frac{P_F(x,n_{.1},n_{1.};n,p_{1.},p_{.1},p_1)}{P_F(x,n_{.1},n_{1.};n,p_{1.},p_{.1},p_0)} = \tag{3.11}$$

$$\frac{p_1^x(p_{1.}-p_1)^{n_{1.}-x}(p_{.1}-p_1)^{n_{.1}-x}(1-p_{1.}-p_{.1}+p_1)^{n-n_{1.}-n_{.1}+x}}{p_0^x(p_{1.}-p_0)^{n_{1.}-x}(p_{.1}-p_0)^{n_{.1}-x}(1-p_{1.}-p_{.1}+p_0)^{n-n_{1.}-n_{.1}+x}}$$

The sequential test is then carried out by using the following procedure:

$$\begin{aligned} &\text{accept } H_0 \text{ if} && Ln_1/Ln_0 \leq B \\ &\text{accept } H_1 \text{ if} && Ln_1/Ln_0 \geq A \\ &\text{take another sample if} && B < Ln_1/Ln_0 < A \end{aligned} \tag{3.12}$$

The values A and B, which are needed for the test, are quite difficult to determine exactly. However, the approximate values

$$A \simeq (1-\beta)/\alpha,\ B \simeq \beta/(1-\alpha) \tag{3.13}$$

given by Wald (1947) are used. Here α is the desired probability of a Type I error and β is the desired probability of a Type II error.

To carry out the test procedure, it is usually more convenient to work in terms of the log likelihood ratio

$$\ell n(Ln_1/Ln_0) = g(x,n_{1.},n_{.1},p_{1.},p_{.1},p_0,p_1) = \tag{3.14}$$

$$x\cdot R_1-(n_{1.}-x)\cdot R_2-(n_{.1}-x)\cdot R_3+(n-n_{1.}-n_{.1}+x)\cdot R_4$$

where $R_1=\ell n(p_1/p_0)$

$R_2=\ell n((p_{.1}-p_1)/(p_{.1}-p_0))$

$R_3=\ell n((p_{1.}-p_1)/(p_{1.}-p_0))$

$R_4=\ell n((1-p_{1.}-p_{.1}+p_1)/(1-p_{1.}-p_{.1}+p_0))$

With this modification, the test procedure becomes

accept H_0 if $\ell n(Ln_1/Ln_0)\leq b$

accept H_1 if $\ell n(Ln_1/Ln_0)\geq a$ (3.15)

take another sample if $b<\ell n(Ln_1/Ln_0)<a$

where $a=\ell n((1-\beta)/\alpha)$ and $b=\ell n(\beta/(1-\alpha))$ and $\ell n(Ln_1/Ln_0)$ is shown in (3.14).

Because the test statistic at each trial is in three dimensions, tables of these test plans will be quite lengthy for large sample sizes. In particular, at each trial n one must specify upper and lower limits on x (the count in cell 1 of Figure 3.3) for each of the $(n+1)^2$ different possible margin arrangements. Another approach, which might be used when a given test will be performed only once, is to compute either the critical limits or the likelihood ratio at each trial in order to decide what action should be taken. The method for finding the critical limits which define the test region is given next.

Letting $c_L(n_{1.},n_{.1},n)$ denote the lower limit and $c_U(n_{1.},n_{.1},n)$ the upper limit for x given the marginal totals

$n_{1.}$ and $n_{.1}$ at trial n, the sequential test procedure becomes

$$\begin{aligned} &\text{accept } H_0 \text{ if} && x \le c_L(n_{1.}, n_{.1}, n) \\ &\text{accept } H_1 \text{ if} && x \ge c_U(n_{1.}, n_{.1}, n) \end{aligned} \tag{3.16}$$

and take another sample if $c_L(n_{1.}, n_{.1}, n) < x < c_U(n_{1.}, n_{.1}, n)$ where x, $n_{1.}$, $n_{.1}$ and n are shown in Figure 3.2. The values $c_L(n_{1.}, n_{.1}, n)$ and $c_U(n_{1.}, n_{.1}, n)$ are easily obtained by inversion of the equations

$$\begin{aligned} b &= g(x, n_{1.}, n_{.1}, n, p_{1.}, p_{.1}, p_0, p_1) = \ell n(Ln_1/Ln_0) \\ a &= g(x, n_{1.}, n_{.1}, n, p_{1.}, p_{.1}, p_0, p_1) = \ell n(Ln_1/Ln_0) \end{aligned} \tag{3.17}$$

by solving for x. These values can be expressed as

$$\begin{aligned} c_L(n_{1.}, n_{.1}, n) &= g^{-1}\left[(b, n_{1.}, n_{.1}, n, p_{1.}, p_{.1}, p_0, p_1)\right] \\ &= \left[(b + n_{1.}(R_2+R_4) + n_{.1}(R_3+R_4) + nR_4)/(R_1+R_2+R_3+R_4))\right] \\ c_U(n_{1.}, n_{.1}, n) &= \left[g^{-1}(a, n_{1.}, n_{.1}, n, p_{1.}, p_{.1}, p_0, p_1)\right] + 1 \\ &= \left[(a + n_{1.}(R_2+R_4) + n_{.1}(R_3+R_4) + nR_4)/(R_1+R_2+R_3+R_4)\right] + 1 \end{aligned} \tag{3.18}$$

where the R_i's are defined in (3.14) and M=[K] is the greatest integer less than or equal to K.

These sequential tests of 2x2 contingency tables can be truncated as indicated in Section 2.4. If the test is truncated at some n, say n_0, one must choose the critical values $c_L(n_{1.}, n_{.1}, n)$ and $c_U(n_{1.}, n_{.1}, n)$ for each of the $(n_0+1)^2$ possible combinations of values which the marginal totals can take on at the truncation trial n_0. Some general rules of thumb are given for doing this; these can be further modified in order to give

the test the desired properties.

As a first guess, the critical values are chosen to be

$$c_L(n_{1.},n_{.1},n)=\left[g^{-1}((a+b)/2,n_{1.},n_{.1},n,p_{1.},p_{.1},p_0,p_1)\right]$$
$$c_U(n_{1.},n_{.1},n)=c_L(n_{1.},n_{.1},n)+1 \qquad (3.19)$$

The value (a+b)/2 is used in an effort to truncate the test while keeping the true α and β errors in the proper proportion.

Any of the $(n_0+1)^2$ values for $c_L(n_{1.},n_{.1},n)$ may be changed (along with $c_U(n_{1.},n_{.1},n)$ such that the second equation in (3.19) holds). Such changes will not affect the ASN function; however, they will change the OC function. Thus, the truncation can be used to "balance" the α and β error probabilities. In order to make the best decision as to which points belong in the acceptance region and which belong in the rejection region at n_0, one can examine the exact probabilities of reaching the different points, (obtained by using the direct method of sequential analysis) under different specified alternate hypotheses.

A numerical example of the above procedure for determining the sequential test regions is now given. Let $p_{i.}=0.5$ and $p_{.1}=0.5$. The hypothesis to be tested is

$$H_0: \ p_{11}=p_0=p_{1.}p_{.1}=0.25 \qquad (3.20)$$
$$\text{versus } H_1: \ p_{11}=p_1=0.40$$

with desired error probabilities $\alpha=0.05$ and $\beta=0.1$. The test is truncated at trial $n_0=25$. For this case, the critical values are

Table 3.1
Critical Values for the
Sequential Test Example

P1= 0.500
P.1= 0.500
P0= 0.250
P1= 0.400

ALPHA= 0.100
BETA= 0.050

TRIAL 5 N.1

N1.	0	1	2	3	4	5
0	-1, 6	-1, 6	0, 6	0, 7	1, 7	1, 8
1	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7
2	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7
3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6
4	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6
5	-2, 3	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5

TRIAL 6 N.1

N1.	0	1	2	3	4	5	6
0	-1, 6	0, 7	0, 7	1, 7	1, 8	2, 8	2, 9
1	-1, 6	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8
2	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8
3	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7
4	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7
5	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6
6	-2, 3	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6

TRIAL 7 N.1

N1.	0	1	2	3	4	5	6	7
0	0, 7	0, 7	1, 8	1, 8	2, 8	2, 9	3, 9	3,10
1	-1, 6	0, 7	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9
2	-1, 6	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9
3	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8
4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8
5	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7
6	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7
7	-2, 3	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6

TRIAL 8 N.1

N1.	0	1	2	3	4	5	6	7	8
0	0, 7	1, 8	1, 8	2, 9	2, 9	3, 9	3,10	4,10	4,11
1	0, 7	0, 7	1, 8	1, 8	2, 8	2, 9	3, 9	3,10	4,10
2	-1, 6	0, 7	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9	3,10
3	-1, 6	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9
4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9
5	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8
6	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8
7	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7
8	-2, 3	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7

TRIAL 9 N.1

N1.	0	1	2	3	4	5	6	7	8	9
0	1, 8	1, 8	2, 9	2, 9	3,10	3,10	4,10	4,11	5,11	5,12
1	0, 7	1, 8	1, 8	2, 9	2, 9	3, 9	3,10	4,10	4,11	5,11
2	0, 7	0, 7	1, 8	1, 8	2, 8	2, 9	3, 9	3,10	4,10	4,11
3	-1, 6	0, 7	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9	3,10	4,10
4	-1, 6	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9	3,10
5	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9
6	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9
7	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8
8	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8
9	-2, 3	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7

TRIAL 10 N.1

N1.	0	1	2	3	4	5	6	7	8	9	10
0	1, 8	2, 9	2, 9	3,10	3,10	4,11	4,11	5,11	5,12	6,12	6,13
1	1, 8	1, 8	2, 9	2, 9	3,10	3,10	4,10	4,11	5,11	5,12	6,12
2	0, 7	1, 8	1, 8	2, 9	2, 9	3, 9	3,10	4,10	4,11	5,11	5,12
3	0, 7	0, 7	1, 8	1, 8	2, 8	2, 9	3, 9	3,10	4,10	4,11	5,11
4	-1, 6	0, 7	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9	3,10	4,10	4,11
5	-1, 6	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9	3,10	4,10
6	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9	3,10
7	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9	3, 9
8	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8	2, 9
9	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8	2, 8
10	-2, 3	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6	0, 6	0, 7	1, 7	1, 8

TRIAL 1 N.1

N1.	0	1
0	-2, 3	-2, 4
1	-2, 3	-2, 3

TRIAL 2 N.1

N1.	0	1	2
0	-2, 4	-2, 4	-2, 5
1	-2, 3	-2, 4	-2, 4
2	-2, 3	-2, 3	-2, 4

TRIAL 3 N.1

N1.	0	1	2	3
0	-2, 4	-2, 5	-1, 5	-1, 6
1	-2, 4	-2, 4	-2, 5	-1, 5
2	-2, 3	-2, 4	-2, 4	-2, 5
3	-2, 3	-2, 3	-2, 4	-2, 4

TRIAL 4 N.1

N1.	0	1	2	3	4
0	-2, 5	-1, 5	-1, 6	0, 6	0, 7
1	-2, 4	-2, 5	-1, 5	-1, 6	0, 6
2	-2, 4	-2, 4	-2, 5	-1, 5	-1, 6
3	-2, 3	-2, 4	-2, 4	-2, 5	-1, 5
4	-2, 3	-2, 3	-2, 4	-2, 4	-2, 5

given in Table 3.1 for each trial up to n=10. These regions were computed using (3.17) and (3.18) and are truncated at trail n_0 using (3.19). They were computed using the computer program listed in the Appendix.

The sequential test is carried out as follows. At each trial an item is selected at random from the population. The presence or absence of each of the two binary characteristics is noted. For the observed marginal totals at trial n, the observed value of x is compared with the proper critical limits in the table (or otherwise computed using (3.19) if no table is available). When one of the critical limits is met, the test is terminated and the proper hypothesis is accepted; otherwise, the test is continued and another observation is taken.

A typical sample for such a test is shown in Table 3.2

Table 3.2
Typical Sequential Sample

TRIAL	D	E	$n_{1.}$	$n_{.1}$	x
1	1	0	0	1	0
2	1	1	1	2	1
3	0	0	1	2	1
4	1	0	1	3	1
5	1	1	2	4	2
6	0	1	3	4	2
7	0	1	4	4	2
8	1	0	4	5	2

Here each inspected item has the characteristic of being either D or $\bar{D}$ and being either E or $\bar{E}$. At trial 10, the observed results are summarized in the table given in Figure 3.4.

	D	$\bar{D}$	
E	2	2	4
$\bar{E}$	3	1	4
	5	3	8

Figure 3.4 Observed 2x2 Contingency Table

Comparing the value x=2 with the proper critical values for the marginal totals $n_{1.}=4$ and $n_{.1}=5$, it is seen that the test is terminated and H_0 is accepted. This sequential test region is evaluated in Section 3.5.

3.4 THEORY FOR SEQUENTIAL TESTS WITH THREE DECISIONS

As explained in Section 2.3, sequential tests for a three decision test procedure can be developed by simultaneously using two SPRTs. The development here uses the same notation and underlying probability model as the last section. For a three decision test procedure the hypotheses are specified as:

$$\begin{aligned} & H_1: \ p_{11}=p_1 \\ \text{versus} \quad & H_0: \ p_{11}=p_0>p_1 \\ \text{versus} \quad & H_2: \ p_{11}=p_2>p_0 \end{aligned} \tag{3.21}$$

In addition, the desired α and β error probabilities are specified (for each hypothesis) along with a truncation point n_0. The test procedure at each trial n involves computing two likelihood ratios,

one for each hypothesis, and comparing them to critical values. The sequential test rules for the test at trial n are:

$$\begin{aligned}
&\text{accept } H_1 \text{ if} && Ln_0/Ln_1 \le B_1 \\
&\qquad\qquad \text{and} && Ln_2/Ln_0 \le B_2, \\
&\text{accept } H_0 \text{ if} && Ln_0/Ln_1 \ge A_1 \\
&\qquad\qquad \text{and} && Ln_2/Ln_0 \le B_2 \\
&\text{accept } H_2 && Ln_0/Ln_1 \ge A_1 \\
&\qquad\qquad \text{and} && Ln_2/Ln_0 \ge A_2,
\end{aligned} \tag{3.22}$$

otherwise, the test is continued by taking another sample and repeating the procedure. Wald's approximations are used to find the values A_1, B_1, A_2 and B_2; that is,

$$\begin{aligned}
A_1 &\simeq (1-\alpha_1)/\beta_1 & A_2 &\simeq (1-\beta_2)/\alpha_2 \\
B_1 &\simeq \alpha_1/(1-\beta_1) & B_2 &\simeq \beta_2/(1-\alpha_2).
\end{aligned} \tag{3.23}$$

The values

$$\begin{aligned}
a_1 &= \ell n(A_1) & a_2 &= \ell n(A_2) \\
b_1 &= \ell n(B_1) & b_2 &= \ell n(B_2)
\end{aligned} \tag{3.24}$$

are used below.

For this case it is again possible, and usually desirable, to compute critical values to be compared with the test statistic at each trial. The minimal sufficient test statistic is again $(x, n_{1.}, n_{.1})$. The use of two SPRTs means that there are four

critical limits for each possible combination of marginal totals at each trial. With observed margin totals $(n_{1.}, n_{.1})$ (see Figure 3.2) at trial n, the test procedure is to

$$\begin{aligned}
&\text{accept } H_1 \text{ if} && x \le c_L(n_{1.}, n_{.1}, n) \\
&&\text{and} \quad & x \le d_L(n_{1.}, n_{.1}, n) \\
&\text{accept } H_0 \text{ if} && x \ge c_U(n_{1.}, n_{.1}, n) \\
&&\text{and} \quad & x \le d_L(n_{1.}, n_{.1}, n) \\
&\text{accept } H_2 \text{ if} && x \ge c_U(n_{1.}, n_{.1}, n) \\
&&\text{and} \quad & x \ge d_L(n_{1.}, n_{.1}, n)
\end{aligned} \tag{3.25}$$

and take another sample if none of these conditions is met. Here $c_L(\cdot)$, $c_U(\cdot)$, $d_L(\cdot)$, $d_U(\cdot)$ are critical limits for SPRT 1 and 2 respectively. The critical limits for the test are computed (using the same notation introduced in Section 3.3) as

for SPRT 1 (3.26)

$$c_L(n_{1.}, n_{.1}, n) = \left[(b_1 + n_{1.}(R_2+R_4) + n_{.1}(R_3+R_4) + nR_4)/(R_1+R_2+R_3+R_4)\right]$$

$$c_U(n_{1.}, n_{.1}, n) = \left[(a_1 + n_{1.}(R_2+R_4) + n_{.1}(R_3+R_4) + nR_4)/(R_1+R_2+R_3+R_4)\right] + 1$$

for SPRT 2

$$d_L(n_{1.}, n_{.1}, n) = \left[(b_2 + n_{1.}(R_2+R_4) + n_{.1}(R_3+R_4) + nR_4)/(R_1+R_2+R_3+R_4)\right]$$

$$d_U(n_{1.}, n_{.1}, n) = \left[(a_2 + n_{1.}(R_2+R_4) + n_{.1}(R_3+R_4) + nR_4)/(R_1+R_2+R_3+R_4)\right] + 1$$

where a_i, b_i, i=1,2 are defined in (3.24) and the other notation is the same as is used in (3.18).

Each SPRT can be truncated separately in the same manner outlined in Section 3.3. The critical limits used in (3.25) can again be computed either individually as the test progresses or in tabular form for the entire test plan. When using a three decision test procedure one must compute two tables, one for each SPRT. The preceding is now illustrated with an extension of the numerical example given in Section 3.3.

Again letting $p_{1.}=p_{.1}=.5$, it is desired to choose among the three hypotheses

$$\begin{aligned} & H_1: \ p_{11}=p_1=0.10 \\ \text{versus } & H_0: \ p_{11}=p_0=p_{1.}p_{1.}=0.25 \\ \text{versus } & H_2: \ p_{11}=p_2=0.40 \end{aligned} \tag{3.27}$$

The desired error probabilities are chosen to be $\alpha_1=\alpha_2=.05$ and $\beta_1=\beta_2=0.1$. The critical limits for the SPRT used to distinguish between H_0 and H_2 are given in the example in Section 3.3. The critical limits to distinguish between H_0 and H_2 are shown in Table 3.3. (Note that the designation of α and β has been reversed beacuse $p_1<p_0$.) A typical sequential sample from the 2x2 table is shown in Table 3.4; the corresponding 2x2 table at trial 10 is shown in Figure 3.5. By examination of both sets of critical values at each trial, one finds that H_0 is accepted at trial 10. The properties of this sequential test region are found in Section 3.6.

Table 3.3

Critical Values for the

Three Decision Sequential Test

P1.= 0.500
P.1= 0.500
P0= 0.100
P1= 0.250

ALPHA= 0.100
BETA= 0.050

TRIAL N1. 5 N.1

	0	1	2	3	4	5
0	3,15	2,14	1,13	0,12	-1,11	-2,10
1	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8
2	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6
3	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4
4	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 2
5	-2, 5	-2, 4	-2, 3	-2, 2	-2, 1	-2, 1

TRIAL N1. 6 N.1

	0	1	2	3	4	5	6
0	5,17	4,16	3,15	2,14	1,13	0,12	-1,11
1	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9
2	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7
3	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5
4	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3
5	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 2	-2, 2
6	-2, 5	-2, 4	-2, 3	-2, 2	-2, 1	-2, 1	-2, 0

TRIAL N1. 7 N.1

	0	1	2	3	4	5	6	7
0	7,19	6,18	5,17	4,16	3,15	2,14	1,13	0,12
1	5,17	4,16	3,15	2,14	1,13	0,12	-1,11	-2,10
2	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8
3	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6
4	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4
5	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 3
6	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 2	-2, 2	-2, 1
7	-2, 5	-2, 4	-2, 3	-2, 2	-2, 1	-2, 1	-2, 0	-2,-1

TRIAL N1. 8 N.1

	0	1	2	3	4	5	6	7	8
0	9,21	8,20	7,19	6,18	5,17	4,16	3,15	2,14	1,13
1	7,19	6,18	5,17	4,16	3,15	2,14	1,13	0,12	-1,11
2	5,17	4,16	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9
3	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7
4	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5
5	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 4
6	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 3	-2, 2
7	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 2	-2, 2	-2, 1	-2, 0
8	-2, 5	-2, 4	-2, 3	-2, 2	-2, 1	-2, 1	-2, 0	-2,-1	-2,-1

TRIAL N1. 9 N.1

	0	1	2	3	4	5	6	7	8	9
0	11,23	10,22	9,21	8,20	7,19	6,18	5,17	4,16	3,15	2,14
1	9,21	8,20	7,19	6,18	5,17	4,16	3,15	2,14	1,13	0,12
2	7,19	6,18	5,17	4,16	3,15	2,14	1,13	0,12	-1,11	-2,10
3	5,17	4,16	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8
4	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6
5	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 5
6	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 4	-2, 3
7	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 3	-2, 2	-2, 1
8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 2	-2, 2	-2, 1	-2, 0	-2,-1
9	-2, 5	-2, 4	-2, 3	-2, 2	-2, 1	-2, 1	-2, 0	-2,-1	-2,-1	-2,-1

TRIAL N1. 10 N.1

	0	1	2	3	4	5	6	7	8	9	10
0	13,25	12,24	11,23	10,22	9,21	8,20	7,19	6,18	5,17	4,16	3,15
1	11,23	10,22	9,21	8,20	7,19	6,18	5,17	4,16	3,15	2,14	1,13
2	9,21	8,20	7,19	6,18	5,17	4,16	3,15	2,14	1,13	0,12	-1,11
3	7,19	6,18	5,17	4,16	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9
4	5,17	4,16	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7
5	3,15	2,14	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 6
6	1,13	0,12	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 5	-2, 4
7	-1,11	-2,10	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 4	-2, 3	-2, 2
8	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 3	-2, 2	-2, 1	-2, 0
9	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3	-2, 2	-2, 2	-2, 1	-2, 0	-2,-1	-2,-1
10	-2, 5	-2, 4	-2, 3	-2, 2	-2, 1	-2, 1	-2, 0	-2,-1	-2,-1	-2,-1	-2,-1

TRIAL N1. 1 N.1

	0	1
0	-2, 7	-2, 6
1	-2, 5	-2, 4

TRIAL N1. 2 N.1

	0	1	2
0	-2, 9	-2, 8	-2, 7
1	-2, 7	-2, 6	-2, 5
2	-2, 5	-2, 4	-2, 3

TRIAL N1. 3 N.1

	0	1	2	3
0	-1,11	-2,10	-2, 9	-2, 8
1	-2, 9	-2, 8	-2, 7	-2, 6
2	-2, 7	-2, 6	-2, 5	-2, 4
3	-2, 5	-2, 4	-2, 3	-2, 2

TRIAL N1. 4 N.1

	0	1	2	3	4
0	1,13	0,12	-1,11	-2,10	-2, 9
1	-1,11	-2,10	-2, 9	-2, 8	-2, 7
2	-2, 9	-2, 8	-2, 7	-2, 6	-2, 5
3	-2, 7	-2, 6	-2, 5	-2, 4	-2, 3
4	-2, 5	-2, 4	-2, 3	-2, 2	-2, 1

Table 3.4
Typical Sequential Sample

TRIAL	D	E	$n_{1.}$	$n_{.1}$	x
1	1	1	1	1	1
2	1	1	2	2	2
3	1	1	3	3	3
4	1	1	4	4	4
5	1	1	5	5	5
6	1	1	6	6	6
7	0	0	6	6	6
8	1	1	7	7	7
9	1	1	8	8	8
10	1	1	9	9	9

	D	$\overline{D}$	
E	9	0	9
$\overline{E}$	0	1	1
	9	1	10

Figure 3.5 Observed 2x2 Contingency Table

3.5 EVALUATION OF THE TWO DECISION TEST REGIONS

This section describes the evaluation of the sequential test plans for 2x2 contingency tables. The direct method of sequential analysis, as outlined in Section 2.2, is used to find the exact values of the important test properties. It will be shown below how to compute the OC function and the distribution of the decisive sample number (DSN). From these, one can also find the ASN function and the true α and β error probabilities, α' and β'. The two decision test procedure obtained in Section 3.3 is evaluated as a numerical example. The results given here are extended in the following section to treat exact evaluation of test plans for a three decision test procedure.

As explained in Section 2.2, the direct method is used by computing both the probability of making each decision and the distribution of the probability remaining in the continuation region at each trial. The probabilities at trial n+1 are computed by convoluting the probability remaining in the continuation region at trial n with the sample taken at trial n+1. This is done for each trial $n=1,2,\ldots n_0$, where n_0 is the truncation trial at which the sequential test is terminated. In order to use the direct method, these probabilities are computed for each possible value of some statistic which is both sufficient and transitive. Transitivity of a statistic S implies that the distribution of S at trial n depends only on the value of S at

trial n-1 and the data observed at trial n. A transitive statistic is necessary to compute the probability of the values of the test statistic from one trial to the next. The minimal sufficient statistic $(x,n_{1.},n_{.1})$ is also (obviously) transitive and is used here to compute the probabilities necessary for the direct method.

Each point in the sample space at trial n can be denoted by $(x,n_{1.},n_{.1})$. From each point $(x,n_{1.},n_{.1})$ which is in the continuation region at trial n, the statistic will take on any one of four values at trial n+1, namely $(x+1,n_{1.}+1,n_{.1}+1)$, $(x,n_{1.},n_{.1}+1)$, $(x,n_{1.}+1,n_{.1})$ or $(x,n_{1.},n_{.1})$ with the probabilities shown in Figure 3.6.

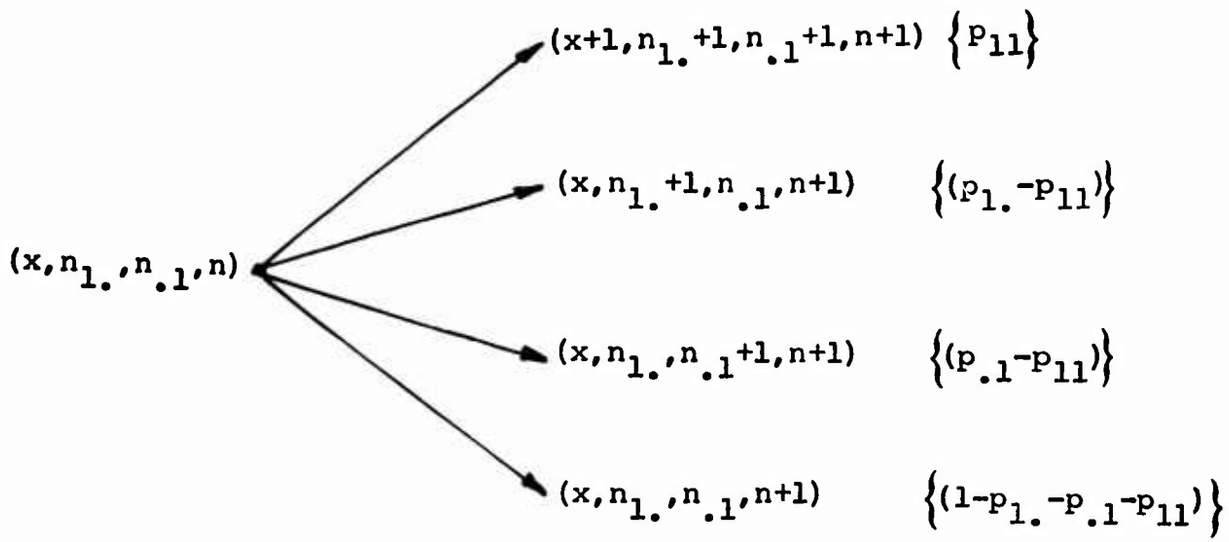

Figure 3.6 Possible Outcomes at Each Trial

The procedure begins at trial 0 where the only possible "outcome" is ($x=0, n_{1.}=0, n_{.1}=0$) which therefore has a probability of 1. The probabilities of reaching each point ($x, n_{1.}, n_{.1}$) at trial n for $n=1,2,\ldots n_0$ are computed recursively starting with this point at the origin.

As shown in Figure 3.6, the probability of reaching each point inside or on the boundary of the sequential test region is a function of the true state of nature. Because the marginal probabilities $p_{1.}$ and $p_{.1}$ are assumed known, the state of nature is completely specified by p_{11} alone. The operating characteristic (OC) and the average sample number (ASN) are functions of the true state of nature and one can specify as many points as necessary or desired at which to evaluate the properties of the sequential test.

After choosing a particular value for the state of nature, the probability of reaching each point in the sample space is computed. This is done by convoluting the probability remaining in the continuation region at trial n with the sample taken at trial n+1. This is done in the following manner.

Let Ai_n denote the event of accepting hypothesis H_i, i=0,1 and C_n the event of being in the continuation region at trial n. That is

$$\begin{aligned} A0_n &= \left\{(x,n_{1.},n_{.1},n) \,|\, x \le c_L(n_{1.},n_{.1},n)\right\} \\ A1_n &= \left\{(x,n_{1.},n_{.1},n) \,|\, x \ge c_U(n_{1.},n_{.1},n)\right\} \\ C_n &= \left\{(x,n_{1.},n_{.1},n) \,|\, c_L(n_{1.},n_{.1},n) < x < c_U(n_{1.},n_{.1},n)\right\} \end{aligned} \tag{3.28}$$

The recursive formula used to find the probabilities for each point in the $(x,n_{1.},n_{.1},n)$ space is:

$$P_S(x,n_{1.},n_{.1},n;\ p_{1.},p_{.1},p_{11})= \tag{3.29}$$

$$I(x-1,n_{1.}-1,n_{.1}-1,n-1)P_S(x-1,n_{1.}-1,n_{.1}-1,n-1;p_{1.},p_{.1},p_{11})p_{11}$$

$$+I(x,n_{1.},n_{.1}-1,n-1)P_S(x,n_{1.},n_{.1}-1,n-1;p_{1.},p_{.1},p_{11})(p_{.1}-p_{11})$$

$$+I(x,n_{1.}-1,n_{.1},n-1)P_S(x,n_{1.}-1,n_{.1},n-1;p_{1.},p_{.1},p_{11})(p_{1.}-p_{11})$$

$$+I(x,n_{1.},n_{.1},n-1)P_S(x,n_{1.},n_{.1},n-1;p_{1.},p_{.1},p_{11})(1-p_{.1}+p_{11})$$

where

$$P_S(x,n_{1.},n_{.1},0;p_{1.},p_{.1},p_{11})= \begin{cases} 1 & \text{if } x=n_{1.}=n_{.1}=0 \\ 0 & \text{otherwise} \end{cases}$$

$$I(x,n_{1.},n_{.1},n)= \begin{cases} 1 & \text{if } (x,n_{1.},n_{.1},n)\in C_n \\ 0 & \text{otherwise} \end{cases}$$

The indicator function I accounts for the fact that the test terminates when one of the critical values is reached. Of course, the probability of all of these points need not be computed; one need only compute the probabilities of those points which are inside or on the boundary of this four-dimensional sequential test region (other points have probability zero).

It should again be noted that the probability of reaching any point $(x,n_{1.},n_{.1},n)$ in the sample space, for a fixed size sample of size n is a multinomial distribution; that is,

$$P_F(x,n_{1.},n_{.1};n,p_{1.},p_{.1},p_{11})= \tag{3.30}$$

$$\frac{n!p_{11}^{x}(p_{1.}-p_{11})^{n_{1.}-x}(p_{.1}-p_{11})^{n_{.1}-x}(1-p_{1.}-p_{.1}+p_{11})^{n-n_{1.}-n_{.1}+x}}{x!(n_{1.}-x)!(n_{.1}-x)!(n-n_{1.}-n_{.1}+x)!}$$

The probability of reaching this point under the sequential test rules, as computed from (3.29), can also be expressed as

$$P_S(x,n_{1.},n_{.1};n,p_{1.},p_{.1},p_{11}) = \tag{3.31}$$
$$K(x,n_{1.},n_{.1},n)p_{11}^{x}(p_{1.}-p_{11})^{n_{1.}-x}(p_{.1}-p_{11})^{n_{.1}-x}(1-p_{1.}-p_{.1}+p_{11})^{x_4}$$

where $x_4=n-n_{1.}-n_{.1}+x$ and $K(x,n_{1.},n_{.1},n)$ is the number of admissable paths to the point $(x,n_{1.},n_{.1},n)$. This leads to a computational simplification when one desires (which is usually the case) to find these probabilities for several or many different values of the true state of nature. If one computes (using (3.29)) the probability of reaching a point $(s,n_{1.},n_{.1},n)$ under the sequential test rules for a specified state of nature p_{11}, the probability of reaching that point under the same sequential test rules, but with true state of nature q_{11} is

$$P_S(x,n_{1.},n_{.1};n,p_{1.},p_{.1},q_{11}) = \tag{3.32}$$
$$\frac{P_S(x,n_{1.},n_{.1};n,p_{1.},p_{.1},p_{11})\ P_F(x,n_{1.},n_{.1};n,p_{1.},p_{.1},q_{11})}{P_F(x,n_{1.},n_{.1};n,p_{1.},p_{.1},p_{11})}$$

$P_F(\cdot)$ is, of course, relatively easy to compute. This simplification is used in the computer programs (which are listed in the Appendix) for finding the sequential test properties.

It is desired to compute the OC function and the distribution of the DSN for different specified values of the state of nature. From these, it is a simple matter to find the ASN function and the true α and β error probabilities.

The probabilities of each of the events $Ai_n, i=0,1$ are computed for each specified state of nature p_{11}. This is done as follows:

$$P(Ai_n, p_{11}) = \tag{3.33}$$

$$\sum_{n_{1.}=0}^{n} \sum_{n_{.1}=0}^{n} \sum_{x=IL}^{IU} J_i(x, n_{1.}, n_{.1}, n)\, P_S(x, n_{1.}, n_{.1}; n, p_{1.}, p_{.1}, p_{11})$$

where $IL = MAX(0, n_{1.} + n_{.1} - n)$

$IU = MIN(n_{1.}, n_{.1})$

$$J_i(x, n_{1.}, n_{.1}, n) = \begin{cases} 1 & \text{if } (x, n_{1.}, n_{.1}, n) \in Ai_n \\ 0 & \text{otherwise} \end{cases}$$

The indicator functions J_i, $i=0,1$ are used to sum only those probabilities which are on the boundary of or outside the sequential test region. Once these probabilities have been computed, the distribution of the DSN can also be computed.

The probability mass function of the DSN is

$$P(n; p_{11}) = P(A0_n \cup A1_n; p_{11}) = P(A0_n; p_{11}) + P(A1_n; p_{11}) \tag{3.34}$$

This is computed up to n_0, the truncation point where $c_U(n_{1.}, n_{.1}, n) = c_L(n_{1.}, n_{.1}, n) + 1$ for all possible combinations of $n_{1.}$ and $n_{.1}$. The ASN function is then computed as

$$ASN(p_{11}) = \sum_{n=1}^{n_0} nP(n; p_{11}) \tag{3.35}$$

Other moments of the distribution of the DSN can also be found. The variance of the DSN is

$$VSN(p_{11})=\sum_{n=1}^{n_0}(n-ASN(p_{11}))^2P(n;p_{11}) \tag{3.36}$$

Similarly, the k^{th} moment about the origin can be expressed as

$$E(n^k;p_{11})=\sum_{n=1}^{n_0}n^kP(n;p_{11}) \tag{3.37}$$

Defining C_n to be the event of being in the continuation region at trial n, the ASN function can also be expressed as

$$ASN(p_{11})=1+\sum_{n=1}^{n_0-1}P(C_n;p_{11}). \tag{3.38}$$

This alternate form is given by Aroian (1975) and shows how the ASN function "builds up" at each trial of the sequential test. The OC function of the sequential test is computed as

$$OC(p_{11})=\sum_{n=1}^{n_0}P(A0_n;p_{11}) \tag{3.39}$$

and the true α and β error probabilities are

$$\begin{aligned}\alpha'&=1-OC(p_0)\\ \beta'&=OC(p_1)\end{aligned} \tag{3.40}$$

The computer program listed in the Appendix finds both the OC and ASN functions for a given sequential test region. Also computed is the probability of continuing to trial n_0 from trial $n_0-1(P(C_{n_0-1}))$. This is the most important point on the CDF (actually one minus the CDF) of the distribution of the DSN and gives the probability that the test will be terminated at trial n_0, the truncation point. In general, this probability

ll be large if the ASN is also large. It is a good measure to help one decide if the sequential test has been truncated too soon.

The test properties have been found for the sequential test region obtained in the numerical example given in Section 3.3. The hypothesis being tested is

$$H_0: \quad p_{11}=p_0=0.25$$

$$\text{versus } H_1: \quad p_{11}=p_1=0.40$$

with desired error probabilities $\alpha=0.05$ and $\beta=0.1$.

These properties were computed using the computer program given in the appendix and are displayed in Table 3.5a. Graphs of the OC and ASN functions are shown in Figure 3.7. It can be seen that the ASN function varies between 8.73 and 15.63 and that the true α and β error probabilities for this sequential test are $\alpha'=0.057$ and $\beta'=0.085$, which are very close to the desired values.

In some cases, the true α and β error probabilities obtained from a given test plan turn out to be different than what is desired (here, $\alpha'=0.057>\alpha=0.05$). In such cases, modification of the test region at the truncation point can be used to achieve the desired error probabilities. Certain points can be moved from the region for acceptance of H_1 to the region for acceptance of H_0. This can be done in a systematic manner by examining the probability of reaching the points in question, under the true states of nature specified by H_0 and H_1 (these probabilities are

Table 3.5a
Test Properties for the
Two Decision Example

$P_{1.}$	$P_{.1}$	P_{11}	$P(H_0)$	$P(H_1)$	ASN	$P(C_{n_0-1})$
0.5	0.5	0.2300	0.97390	0.02609	8.73	0.03671
0.5	0.5	0.2400	0.96084	0.03915	9.41	0.05196
0.5	0.5	0.2500	0.94278	0.05722	10.13	0.07091
0.5	0.5	0.2600	0.91855	0.08144	10.89	0.09345
0.5	0.5	0.2700	0.88709	0.11291	11.67	0.11900
0.5	0.5	0.2800	0.84750	0.15250	12.44	0.14652
0.5	0.5	0.2900	0.79925	0.20075	13.19	0.17450
0.5	0.5	0.3000	0.74234	0.25765	13.89	0.20097
0.5	0.5	0.3200	0.60577	0.39423	15.02	0.24061
0.5	0.5	0.3400	0.45103	0.54897	15.63	0.24941
0.5	0.5	0.3600	0.29959	0.70040	15.58	0.22026
0.5	0.5	0.3700	0.23222	0.76777	15.30	0.19333
0.5	0.5	0.3800	0.17339	0.82661	14.87	0.16112
0.5	0.5	0.3900	0.12434	0.87565	14.30	0.12666
0.5	0.5	0.4000	0.08542	0.91457	13.63	0.09309
0.5	0.5	0.4100	0.05608	0.94391	12.88	0.06323
0.5	0.5	0.4200	0.03510	0.96489	12.10	0.03905
0.5	0.5	0.4300	0.02087	0.97912	11.31	0.02144

Table 3.5b
Test Properties for the
Two Decision Example
(Favoring H_0)

$P_{1.}$	$P_{.1}$	P_{11}	$P(H_0)$	$P(H_1)$	ASN	$P(C_{n_0-1})$
0.5	0.5	0.2300	0.98415	0.01585	8.73	0.03671
0.5	0.5	0.2400	0.97672	0.02328	9.41	0.05196
0.5	0.5	0.2500	0.96640	0.03360	10.13	0.07091
0.5	0.5	0.2600	0.95237	0.04763	10.89	0.09345
0.5	0.5	0.2700	0.93370	0.06630	11.67	0.11900
0.5	0.5	0.2800	0.90939	0.09060	12.44	0.14652
0.5	0.5	0.2900	0.87849	0.12150	13.19	0.17450
0.5	0.5	0.3000	0.84013	0.15986	13.89	0.20097
0.5	0.5	0.3200	0.73893	0.26107	15.02	0.24061
0.5	0.5	0.3400	0.60613	0.39386	15.63	0.24941
0.5	0.5	0.3600	0.45168	0.54832	15.58	0.22026
0.5	0.5	0.3700	0.37226	0.62774	15.30	0.19333
0.5	0.5	0.3800	0.29545	0.70455	14.87	0.16112
0.5	0.5	0.3900	0.22439	0.77561	14.30	0.12666
0.5	0.5	0.4000	0.16185	0.83814	13.63	0.09309
0.5	0.5	0.4100	0.10987	0.89012	12.88	0.06323
0.5	0.5	0.4200	0.06942	0.93058	12.10	0.03905
0.5	0.5	0.4300	0.04027	0.95973	11.31	0.02144

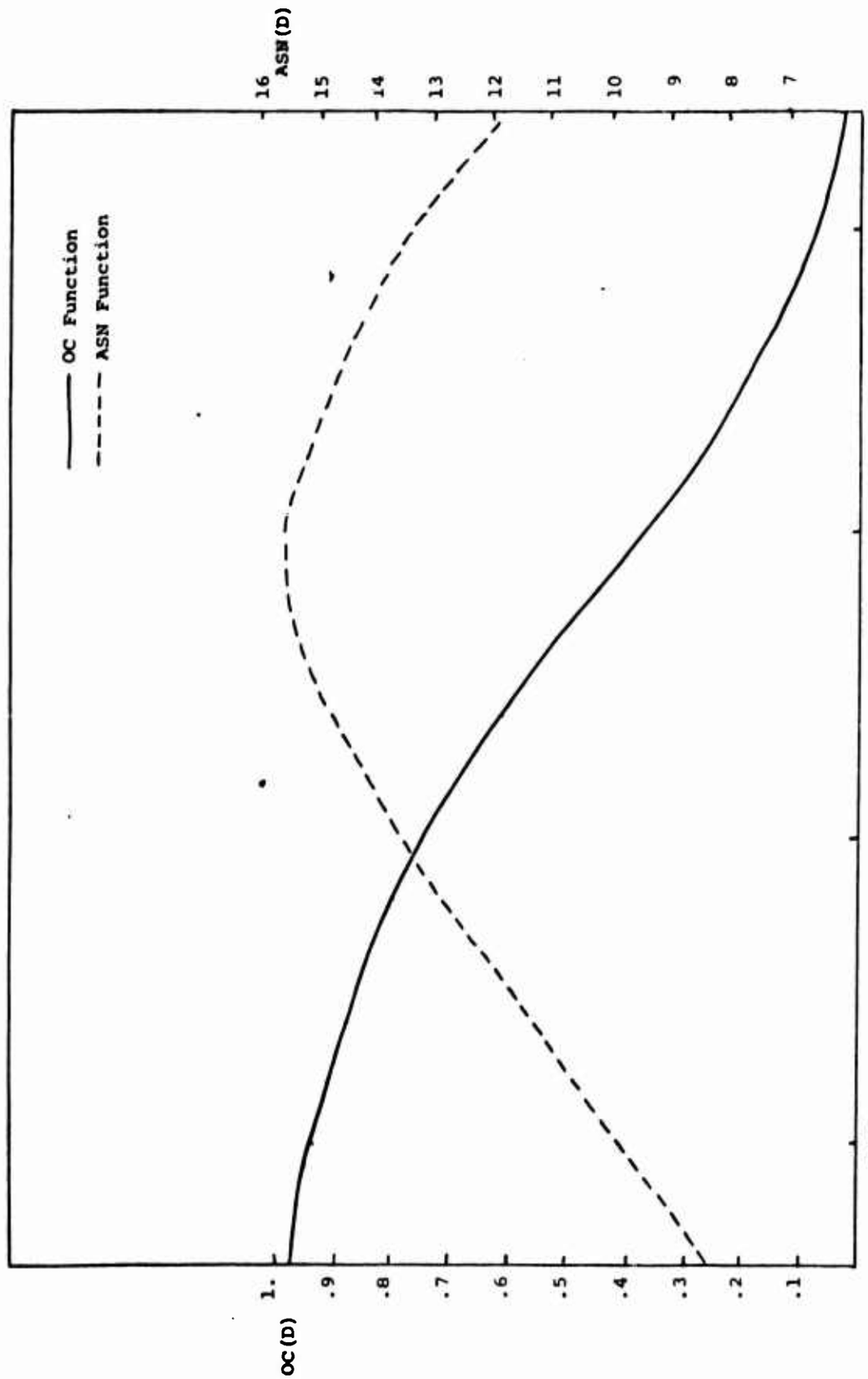

Figure 3.7 Graphs of the OC and ASN Functions for the Two Decision Example

obtained by using the direct method of sequential analysis). This type of modification will of course result in some loss of power. Table 3.5b shows the test properties for the previous numerical example with the region modified in this manner. It is seen that α' is reduced from 0.057 to 0.034 and that the power $(1-\beta')$ is reduced from 0.915 to 0.838. It should be noted that the ASN function and $P(C_{n_0-1})$ remain the same for such modifications. Such procedures for region modification are used in succeeding numerical examples and are treated more fully in Chapter 7.

In order to show the relative superiority of this sequential procedure, the above results are now compared, for the one-sided test procedure, with a similar fixed size sample test. The fixed size test with sample size $n^*=20$ is used. The critical region (for rejection of H_0) for this test was found by including in it all of the points which favor H_1 and have the smallest probabilities summing to 0.057, the true α error probability of the sequential test. The power function for this fixed size test is shown in Table 3.6. It is seen from this that the sequential test has both higher power and an ASN function which is uniformly less than the fixed size sample number, $n^*=20$.

Table 3.6
Power Function for the Fixed Size
Sample Test ($n^*=20$)

p_{11}	$P(H_1;p_{11})$
.2	.0136
.21	.0186
.22	.0250
.23	.0333
.24	.0438
.25	.0571
.26	.0737
.27	.0942
.28	.1193
.29	.1495
.3	.1854
.31	.2274
.32	.2757
.33	.3300
.34	.3902
.35	.4552
.36	.5238
.37	.5942
.38	.6644
.39	.7322
.4	.7952
.41	.8513
.42	.8987
.43	.9363
.44	.9640
.45	.9823

3.6 EVALUATION OF THE THREE DECISION TEST REGIONS

This section describes the method whereby one can obtain the test properties of the three decision sequential test regions found in Section 3.4. The evaluations performed here are similar to those in the previous section; the preliminary information presented there is not repeated here. Again, the OC function and distribution of the DSN are found, from which one can easily obtain the ASN function and the true α and β error probabilities.

At each trial, an observation is taken and either one of the three hypotheses is accepted, terminating the test, or the test is continued by taking another observation. This can be continued up to trial n_0, the truncation point.

Let Ai_n denote the events of accepting hypotheses H_i at trial n, i= 0,1,2 and C_n the event of continuing to trial n+1. (Note: $P(C_0)=1$, $P(C_{n_0})=0$.) There will be an OC function associated with each of the three hypotheses giving the probability of accepting H_i under a specified state of nature p_{11}. Each point in the sample space can again be denoted $(x,n_{1.},n_{.1},n)$. Each of these points is a member of one of the above-mentioned sets, that is,

$$\begin{aligned} A1_n &= \{(x,n_{1.},n_{.1},n) \mid x \le c_L(n_{1.},n_{.1},n) \text{ and } x \le d_L(n_{1.},n_{.1},n)\} \\ A0_n &= \{(x,n_{1.},n_{.1},n) \mid x \ge c_U(n_{1.},n_{.1},n) \text{ and } x \le d_L(n_{1.},n_{.1},n)\} \\ A2_n &= \{(x,n_{1.},n_{.1},n) \mid x \ge c_U(n_{1.},n_{.1},n) \text{ and } x \ge d_U(n_{1.},n_{.1},n)\} \end{aligned} \tag{3.41}$$

or the continuation region, C_n.

Again, it is necessary to find the probability of reaching each point in the sample space under the specified sequential test rules and different states of nature; that is, $P_S(x,n_{1.},n_{.1},n)$

Using the same general procedure outlined in the last section, these probabilities are found recursively using the following formula

$$P_S(x,n_{1.},n_{.1};p_{1.},p_{.1},p_{11})= \tag{3.42}$$

$$I(x-1,n_{1.}-1,n_{.1}-1,n-1)P_S(x-1,n_{1.}-1,n_{.1}-1,n-1;p_{1.},p_{.1},p_{11})p_{11}$$

$$+I(x,n_{1.}-1,n_{.1},n-1)P_S(x,n_{1.}-1,n_{.1},n-1;p_{1.},p_{.1},p_{11})(p_{1.}-p_{11})$$

$$+I(x,n_{1.},n_{.1}-1,n-1)P_S(x,n_{1.},n_{.1}-1,n-1;p_{1.},p_{.1},p_{11})(p_{.1}-p_{11})$$

$$+I(x,n_{1.},n_{.1},n-1)P_S(x,n_{1.},n_{.1},n-1;p_{1.},p_{.1},p_{11})(1-p_{1.}-p_{.1}+p_{11})$$

where

$$P_S(x,n_{1.},n_{.1},0;p_{1.},p_{.1},p_{11})= \begin{cases} 1 & \text{if } x-n_{1.}=n_{.1}=0 \\ 0 & \text{otherwise} \end{cases}$$

and $$I(x,n_{1.},n_{.1},n-1)= \begin{cases} 0 & \text{if } (x,n_{.1},n_{.1},n-1)\in C_n \\ 1 & \text{otherwise} \end{cases}$$

Here the indicator function I accounts for the fact that the test terminates when the test statistic leaves the continuation region. The simplification for computation of these probabilities given in the last section is also applicable here.

The probability of each of the events Ai_n, i=0,1,2 is computed for each trial $n=1,2,\ldots,n_0$. This is done as follows

$$P(Ai_n,p_{11})=$$

$$\sum_{n_{1.}=0}^{n_0}\sum_{n_{.1}=0}^{n_0}\sum_{x=IL}^{IU} J_i(x,n_{1.},n_{.1},n)P_S(x,n_{1.},n_{.1},n;p_{1.},p_{.1},p_{11})$$

$$\text{where} \quad J_i=\begin{cases}1 & \text{if } (x,n_{1.},n_{.1},n)\in Ai_n\\ 0 & \text{otherwise}\end{cases} \tag{3.43}$$

$$\text{and} \quad IL=MAX(0,n_{1.}+n_{.1}-n)$$

$$IU=MIN(n_{1.},n_{.1})$$

The probability mass function of the DSN can be expressed as

$$P(n;p_{11})=P(\bigcup_{i=0}^{2}Ai_n)=\sum_{i=0}^{2}P(Ai_n;p_{11}) \tag{3.44}$$

and is computed for $n=1,2,\ldots,n_0$. The ASN function is then

$$ASN(p_{11})=\sum_{n=1}^{n_0}nP(n;p_{11}) \tag{3.45}$$

Other moments can similarly be expressed as in (3.37).

The OC function of the i^{th} hypothesis gives the probability of accepting that hypothesis as a function of the true state of nature and is computed as

$$OC_i(p_{11})=\sum_{n=1}^{n_0}P(Ai_n,p_{11}) \tag{3.46}$$

The true α and β error probabilities for each SPRT are found as

$$\begin{aligned}\alpha_1'&=OC_1(p_0) & \beta_1'&=OC_0(p_1)\\ \alpha_2'&=OC_2(p_0) & \beta_2'&=OC_0(p_2)\end{aligned} \tag{3.47}$$

The above properties, along with the probability of continuation to trial n_0, are computed by the computer program listed in the Appendix.

For the numerical example concerning the three decision test given at the end of Section 3.4, the hypotheses being tested are

$$\begin{aligned} H_1&: \ p=p_1=0.10 \\ \text{versus } H_0&: \ p=p_0=0.25 \\ \text{versus } H_2&: \ p=p_2=0.40 \end{aligned} \tag{3.47}$$

with desired error probabilities $\alpha_1=\alpha_2=0.05$ and $\beta_1=\beta_2=0.1$. The exact test properties for this example are given in Table 3.7a. The OC and ASN functions are graphed in Figure 3.8. Table 3.7b shows the test properties for the same sequential test region, using the same truncation modification described in Section 3.5.

Table 3.7a
Test Properties for the
Three Decision Example

$P_1.$	$P_{.1}$	P_{11}	$P(H_1)$	$P(H_0)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.5	0.5	0.0500	0.99901	0.00099	0.00000	9.80	0.00436
0.5	0.5	0.0800	0.98367	0.01632	0.00000	12.28	0.04395
0.5	0.5	0.1000	0.94501	0.05497	0.00002	14.11	0.10509
0.5	0.5	0.1200	0.86642	0.13350	0.00007	15.81	0.18249
0.5	0.5	0.1500	0.66980	0.32974	0.00045	17.56	0.27289
0.5	0.5	0.1800	0.42817	0.56934	0.00248	18.06	0.27889
0.5	0.5	0.2000	0.28224	0.71071	0.00704	17.87	0.24215
0.5	0.5	0.2300	0.12496	0.84690	0.02814	17.41	0.18001
0.5	0.5	0.2400	0.09038	0.86727	0.04234	17.33	0.16872
0.5	0.5	0.2500	0.06365	0.87433	0.06201	17.30	0.16519
0.5	0.5	0.2600	0.04364	0.86797	0.08838	17.35	0.16991
0.5	0.5	0.2700	0.02913	0.84827	0.12259	17.45	0.18238
0.5	0.5	0.3000	0.00740	0.71349	0.27911	17.98	0.24781
0.5	0.5	0.3200	0.00262	0.57233	0.42504	18.22	0.28613
0.5	0.5	0.3500	0.00047	0.33201	0.66751	17.79	0.28057
0.5	0.5	0.3800	0.00007	0.13455	0.86537	16.07	0.18795
0.5	0.5	0.4000	0.00002	0.05541	0.94457	14.35	0.10836
0.5	0.5	0.4200	0.00000	0.01645	0.98355	12.47	0.04537
0.5	0.5	0.4500	0.00000	0.00099	0.99900	9.89	0.00451

Table 3.7b
Test Properties for the
Three Decsion Example
(Favoring H_0)

$P_1.$	$P_{.1}$	P_{11}	$P(H_1)$	$P(H_0)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.5	0.5	0.0500	0.99488	0.00512	0.00000	9.80	0.00436
0.5	0.5	0.0800	0.94521	0.05478	0.00000	12.28	0.04395
0.5	0.5	0.1000	0.85923	0.14074	0.00002	14.11	0.10509
0.5	0.5	0.1200	0.72928	0.27065	0.00007	15.81	0.18249
0.5	0.5	0.1500	0.49296	0.50665	0.00039	17.56	0.27289
0.5	0.5	0.1800	0.27813	0.72005	0.00181	18.06	0.27889
0.5	0.5	0.2000	0.17198	0.82344	0.00458	17.87	0.24215
0.5	0.5	0.2300	0.07255	0.91144	0.01621	17.41	0.18001
0.5	0.5	0.2400	0.05220	0.92393	0.02386	17.33	0.16872
0.5	0.5	0.2500	0.03697	0.92851	0.03451	17.30	0.16519
0.5	0.5	0.2600	0.02571	0.92527	0.04901	17.35	0.16991
0.5	0.5	0.2700	0.01755	0.91409	0.06835	17.45	0.18238
0.5	0.5	0.3000	0.00500	0.82962	0.16537	17.98	0.24781
0.5	0.5	0.3200	0.00197	0.72780	0.27022	18.22	0.28613
0.5	0.5	0.3500	0.00042	0.51463	0.48494	17.79	0.28057
0.5	0.5	0.3800	0.00007	0.27622	0.72371	16.07	0.18795
0.5	0.5	0.4000	0.00002	0.14405	0.85593	14.35	0.10836
0.5	0.5	0.4200	0.00000	0.05621	0.94378	12.47	0.04537
0.5	0.5	0.4500	0.00000	0.00527	0.99473	9.89	0.00451

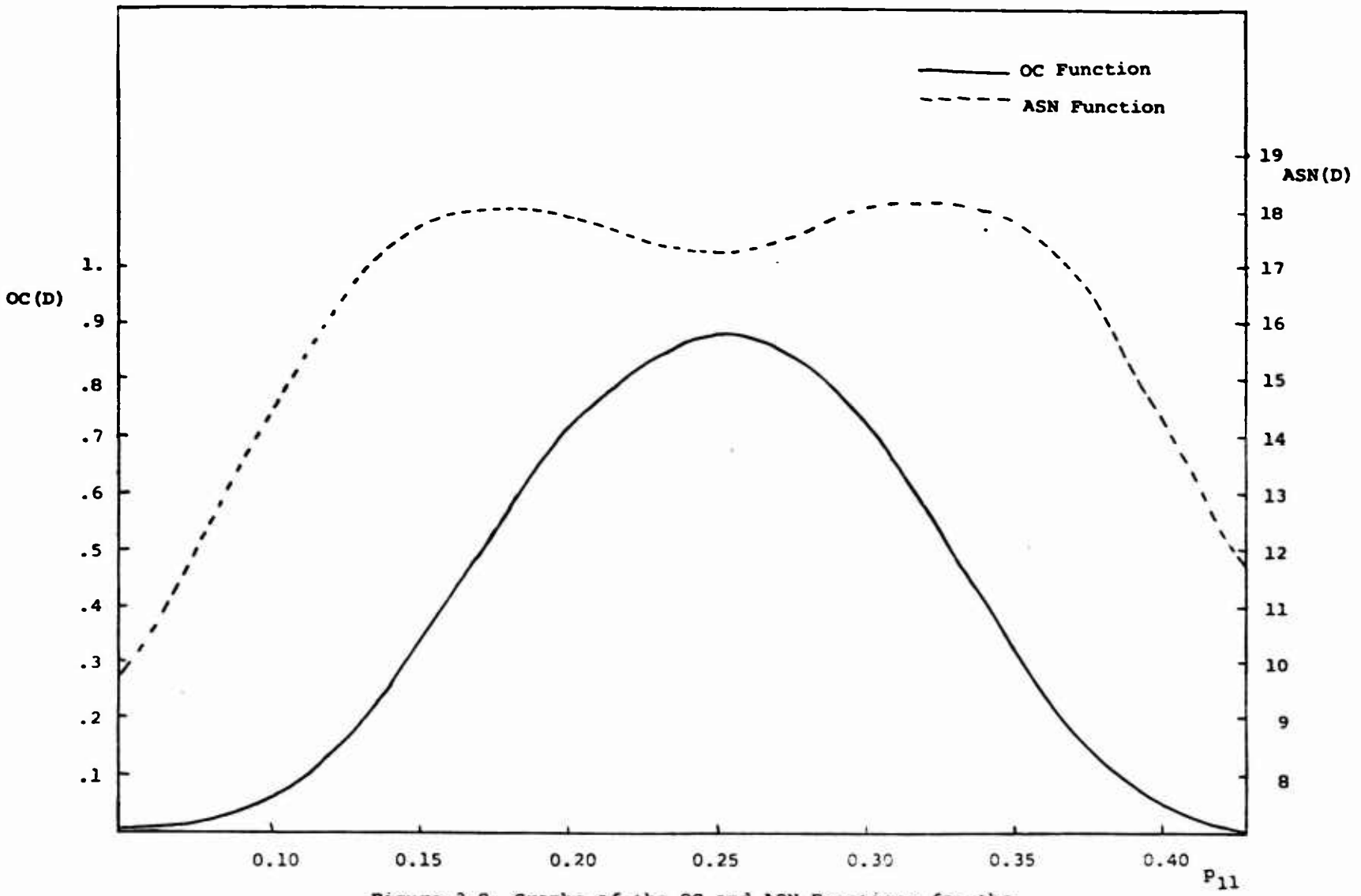

Figure 3.8 Graphs of the OC and ASN Functions for the Three Decision Example

CHAPTER 4

SEQUENTIAL TESTS WHEN THE MARGINAL PROBABILITIES ARE UNKNOWN

4.0 INTRODUCTION

This chapter treats sequential methods for testing 2x2 contingency tables when the marginal probabilities are unknown. Section 4.1 discusses these tables and the hypothesis being tested. Also introduced is the cross product ratio, the parameter on which the sequential tests are based. Section 4.2 examines Fisher's exact test for 2x2 tables with small samples, along with the related "extended hypergeometric distribution." Section 4.3 develops the theory for the construction of the sequential test regions for both two and three decision test procedures. The last section shows how the exact properties of these regions can be determined. Numerical examples are also given.

4.1 THE HYPOTHESIS BEING TESTED AND THE CROSS PRODUCT RATIO

The underlying probability model for the 2x2 contingency table treated here is the same as that given in Figure 3.1, except that in the case considered here, the marginal probabilities $p_{1.}$ and $p_{.1}$ are assumed to be unknown. This is the case termed the "double dichotomy" by Barnard (1947a) and discussed in Section 1.3. The hypothesis being tested is for

independence or for some specified degree of dependence between the two marginal distributions. As indicated in Section 3.2, this can be expressed in terms of several different parameters. The most convenient parameter to use for the present case is the cross product ratio (CPR)

$$t = \frac{p_{11}(1-p_{1.}-p_{.1}+p_{11})}{(p_{1.}-p_{11})(p_{.1}-p_{11})} \qquad (4.1)$$

The cross product ratio has a long history in the analysis of contingency table data for which it has been used as a measure of association. When t=1, the two marginal distributions are independent. For t>1 there is negative dependence, and for t<1, there is positive dependence between the marginal distributions. The hypotheses for tests of independence can be expressed as

$$H_0: \quad t=t_0=1$$
$$\text{versus } H_1: \quad t=t_1\neq 1 \qquad (4.2)$$

The cross product ratio is only one of many measures of association which have been proposed. The papers of Goodman and Kruskal (1954, 1959, 1963 and 1971) discuss many of these. Some are functions of the χ^2 statistic; others are functions of a difference of probabilities or of ratios of probabilities. Most authors, however, agree that the cross product ratio, as a measure of association, has most of the desirable characteristics, certainly more than most other measures which have been proposed for use with 2x2 tables. This point is made, for example, by Fleiss (1973) and Edwards (1963); the latter asserts that the measure of association in a 2x2 contingency "should

logically be some function of the cross-ratio." Fleiss also mentions some criticism of the cross product ratio, first pointed out by Berkson (1958). That is, the level of each rate is lost in computing the ratio; this, however is true for almost all measures of association.

It seems that the CPR first appeared as the parameter of interest in Fisher's exact test (Fisher, 1935) for 2x2 tables, as discussed in the next section. Wald uses the closely related odds ratio for comparing two unknown binomial properties. The odds ratio between two binomial probabilities is

$$t = \frac{p_1(1-p_2)}{p_2(1-p_1)} \qquad (4.3)$$

Such tests are discussed in the next chapter. The odds ratio is also the parameter which is used to specify the hypotheses in Girshick's sequential two sample tests for Darmois-Koopman type populations (Girshick, 1946). Contours for the odds ratio with respect to p_1 and p_2 are shown in Figure 4.1.

Cornfield (1956b) uses the cross product ratio in retrospective studies. Fleiss (1973) discusses this and further explains the invariance of the measure to different types of studies. This type of invariance is an important advantage of the CPR. That is, if researchers are studying a phenomenon using different methods (e.g., retrospective versus prospective studies) the measure of association being studied will, on the average, be the same for the different studies. Also, the odds ratio is the natural parameter of association when a logistic model is used.

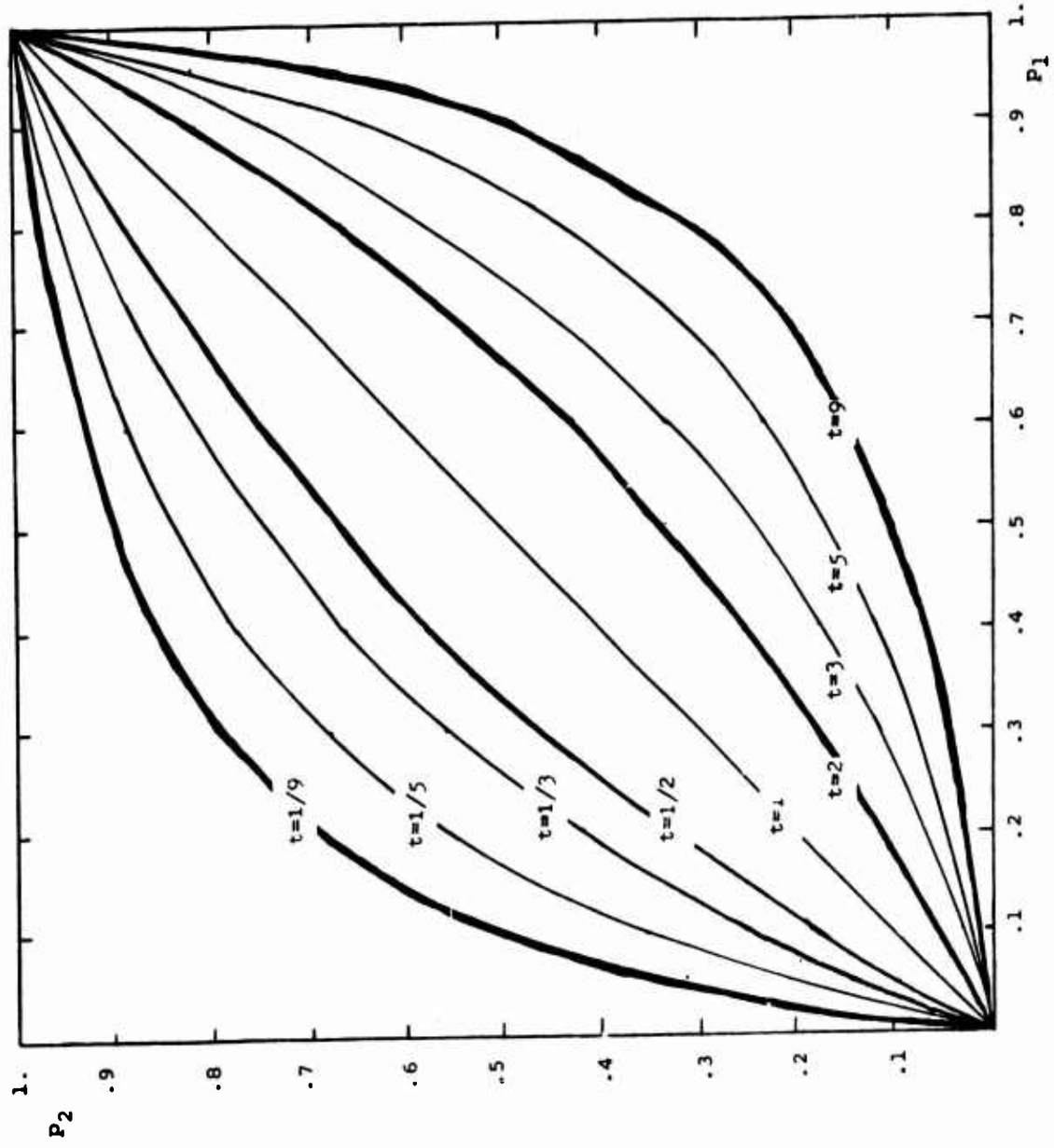

Figure 4.1 Odds Ratio

The logistic form is treated by Cox (1958) and Gart (1971), and is briefly discussed in the next chapter.

Estimates and confidence limits for the CPR and odds ratio can also be found. Methods for doing this are given, for example, by Fisher (1962), Goodman (1964) and Harkness (1959). A computer algorithm for finding such estimates and confidence limits is given by Thomas (1971). Sequential tests of hypotheses concerning this parameter are treated in this chapter.

Because inferences are to be made on the cross product ratio or the odds ratio, rather than the actual probabilities in the table, the marginal probabilities are so-called nuisance parameters. That is, their values give no information concerning the inferences to be made, but they do affect the overall power of the procedures used.

The following is a justification for the use of the cross product ratio when making comparisons between probabilities. This short discussion concerns the comparison of two unknown binomial proportions; the ideas presented, however, are useful when testing for independence in the 2x2 contingency tables considered here. When comparing two binomial probabilities, one might consider using the difference between the probabilities

$$\Delta = p_1 - p_2$$

to measure the degree of inequality. This is a rather poor measure, however, because the importance of a given value of Δ depends on the actual magnitudes of p_1 and p_2. For example, the

with small samples. This fixed size test is developed and discussed here. The test is to be used with the probability model shown in Figure 3.1, with unknown marginal probabilities $p_{1.}$ and $p_{.1}$. The probability of observing the 2x2 table shown in Figure 3.2 is then the multinomial distribution

$$P_F(x,n_{1.},n_{.1};n,p_{1.},p_{.1},p_{11}) = \tag{4.5}$$

$$\frac{n!\, p_{11}^{x}(p_{1.}-p_{.1})^{n_{1.}-x}(p_{.1}-p_{11})^{n_{.1}-x}(1-p_{1.}-p_{.1}+p_{11})^{n-n_{1.}-n_{.1}+x}}{x!(n_{1.}-x)!(n_{.1}-x)!(n-n_{1.}-n_{.1}+x)!}$$

The hypothesis of independence to be tested is expressed in terms of the cross product ratio

$$t = \frac{p_{11}(1-p_{1.}-p_{.1}+p_{11})}{(p_{1.}-p_{11})(p_{.1}-p_{11})} . \tag{4.6}$$

When t=1, the hypothesis of independence is true. As the parameters $p_{1.}$ and $p_{.1}$ are unknown, they are nuisance parameters having no direct bearing on the degree of association.

Thus, Fisher's exact test is conditional on the observed marginal totals. The conditional distribution is independent of these nuisance parameters. The probability of observing the sample table shown in Figures 3.2, conditioned on the observed marginal totals, $n_{1.}$ and $n_{.1}$ is then easily shown to be

$$P_C(x;n_{1.},n_{.1},n,t) = \frac{\binom{n_{1.}}{x}\binom{n-n_{1.}}{n_{.1}-x} t^{x}}{\sum_{j=IL}^{IU}\binom{n_{1.}}{j}\binom{n-n_{1.}}{n_{.1}-j} t^{j}} \tag{4.7}$$

where $IL=MAX(0,n_{1.}+n_{.1}-n)$

$IU=MIN(n_{1.},n_{.1})$

and t is the cross product ratio. For the case of independence, when t=1, (4.7) reduces to

$$P_C(x,n_{1.},n_{.1},n,1)=\frac{\binom{n_{1.}}{x}\binom{n-n_{1.}}{n_{.1}-x}}{\binom{n}{n_{.1}}} \qquad (4.8)$$

This is simply the hypergeometric distribution for which tables of the probability mass and cumulative distribution functions are given by Lieberman and Owen (1961).

Fisher's exact test for independence is conducted by choosing as the critical region (for each of the different combinations of $n_{1.}$ and $n_{.1}$) those values of x which have the smallest probabilities (in one or both tails of the conditional distribution) summing to the desired significance level under the null distribution in (4.8). Tables for such tests are given, for example, by Armsen (1955) and Owen (1962).

As mentioned earlier, there have been some arguments with this approach. The controversy arises because the test is conditioned on the observed margins, greatly reducing the reference set from which the critical region is chosen. Fisher's argument (Fisher, 1935) in favor of this approach is based on the theory of sufficient and ancillary statistics. Because $n_{1.}$ and $n_{.1}$ provide no information about the degree of dependence (i.e., t) they are ancillary statistics. Ancillary statistics do, however,

indicate the amount of information concerning the degree of dependence which is available from the sample. Inferences about t, the CPR, should therefore be made conditional on the ancillary statistics. Lehmann (1959) shows that the uniformly most powerful unbiased tests of hypotheses concerning t must be based on the conditional distribution of x given $(n_{1.}, n_{.1})$. This argument is given a more rigorous treatment in Section 5.2 where the problem is presented in the logistic form.

The distribution in (4.7) is known as the "extended hypergeometric distribution." This distribution gives the probability of observing a given 2x2 contingency table, conditional on the observed margins, for any value of t, the CPR. Harkness (1965) discusses this distribution and its properties in detail.

While the conditional distribution in (4.7) is useful for testing hypotheses about t, the unconditional multinomial distribution in (4.5) must be used to find the power of the test. Harkness (1959) and Harkness and Katz (1964) treat the power of the uniformly most powerful unbiased test (UMPUT) discussed, for example, by Lehmann (1959). They also compare the power function of the different 2x2 table models outlined in Section 1.3. The power of the UMPUT test is compared with sequential tests presented later. Following Lehmann (1959), the UMPUT of size α for

$$H_0: \ t=t_0 \qquad (4.9)$$

versus $H_1: \ t \neq t_0$

is

$$\phi(n_{1.},n_{.1},x)= \begin{cases} 1 & \text{if } x<c_1(n_{1.},n_{.1}) \text{ or } x>c_2(n_{1.},n_{.1}) \\ \gamma_i & \text{if } x=c_i(n_{1.}n_{.1}),\ i=1,2 \\ 0 & \text{if } c_1(n_{1.},n_{.1})<x<c_2(n_{1.},n_{.1}) \end{cases} \qquad (4.10)$$

where c_i and γ_i are values satisfying the equations

$$E(\phi(n_{1.},n_{.1},x))= \alpha E(x\phi(n_{1.},n_{.1},x))= \alpha E(x) \qquad (4.11)$$

and the expectations are taken with respect to the null distribution, that is, the hypergeometric in (4.8). This is a randomized version of Fisher's test enabling the probability of a Type I error to be exactly α. This test is compared with the sequential test in Section 4.4.

In the examination of the power function of the UMPUT given by Harkness (1959), it is important to note that the α error in all cases has a value of 0.05. That is, the probability of rejecting H_0 when t=1 is 0.05. The power, however, varies considerably over equal values of $t\neq 1$, depending on the values of the nuisance parameters $p_{1.}$ and $p_{.1}$. Thus the power (with respect to t) of the test is dependent on these nuisance parameters. The power is greatest when $p_{1.}$ and $p_{.1}$ are near 0.5. The reduction of power for the more extreme values of $p_{1.}$ and $p_{.1}$ will also occur to a lesser extent in the truncated sequential test developed here. This will be discussed further with the examination of the exact test properties here and in Chapters 5 and 7.

4.3 THEORY FOR SEQUENTIAL TESTS WITH TWO AND THREE DECISIONS

This section develops the theory for sequentially testing the hypothesis of independence of 2x2 contingency tables. It is assumed that both marginal totals are random variables with unknown probability distributions. The tests are based on the extended hypergeometric distribution and the minimal sufficient statistic $(x,n_{1.},n_{.1},n)$ from the table in Figure 3.2. The underlying probability model is the same as shown in Figure 3.1, with $p_{1.}$ and $p_{.1}$ now assumed unknown.

To test the hypothesis

$$\begin{aligned} &H_0: \quad t=t_0 \\ \text{versus } &H_1: \quad t=t_1 \neq t_0 \end{aligned} \tag{4.12}$$

a Wald-type SPRT can be constructed by using the ratio

$$Ln_1/Ln_0 = \frac{P_C(x,n_{1.},n_{.1},n,t_1)}{P_C(x,n_{1.},n_{.1},n,t_0)} \tag{4.13}$$

where $P_C(\cdot)$ is the conditional distribution in (4.7). The rules for the sequential test procedure are then to

$$\begin{aligned} &\text{accept } H_0 \quad \text{if } \ell n(Ln_1/Ln_0) \leq b \\ &\text{accept } H_1 \quad \text{if } \ell n(Ln_1/Ln_0) \geq a. \end{aligned} \tag{4.14}$$

Otherwise the test is continued and another sample is taken. Here a and b are again approximated by the values

$$\begin{aligned} a &\simeq \ell n(A) = \ell n(\beta/(1-\alpha)) \\ b &\simeq \ell n(B) = \ell n((1-\beta)/\alpha) \end{aligned} \tag{4.15}$$

The ratio in (4.13) does not represent a likelihood ratio in the true sense of the word; it is a probability ratio, conditional on the observed values of the ancillary statistics. The values of t_0 and t_1 to be used for the test can be chosen with the aid of Figure 4.1. Using the argument of ancillary statistics put forward by Fisher, the ratio in (4.13) is a logical method of determining critical values for the sequential tests presented here.

Paulson (1970) suggests a conditional sequential test, for two sample problems of the Darmois-Koopman form, which is conditional on an ancillary statistic. He rejects the formulation, however, because the test properties are "difficult to determine." He then suggests a test based on the ratio of moment generating functions which would be guaranteed to meet the specified error probabilities. By using the direct method of sequential analysis, however, one can find the exact properties of any such sequential tests, as shown below.

Although the individual critical values will in general be different, the sequential test regions will take on the same form as the tests presented for the case of known marginal probabilities treated in Chapter 3. At each trial $n=1,2,\ldots n_0$, there are again two critical values for x, $c_L(n_{1.},n_{.1},n)$ and $c_U(n_{1.},n_{.1},n)$, for each of the $(n+1)^2$ different combinations of the marginal totals. The critical values $c_L(n_{1.},n_{.1},n)$ and $c_U(n_{1.},n_{.1},n)$ have the same meaning here as illustrated by the sequential test rules shown in (3.16). The critical values for the present case are found by inverting the log likelihood ratio equations

$$b=g(x,n_{1.},n_{.1},n,t_0,t_1)=\ell n(Ln_1/Ln_0)$$
$$a=g(x,n_{1.},n_{.1},n,t_0,t_1)=\ell n(Ln_1/Ln_0) \tag{4.16}$$

again by solving for x. These values can be expressed as

$$\begin{aligned} c_L(n_{1.},n_{.1},n) &= \left[g^{-1}(b,n_{1.},n_{.1},n,t_0,t_1) \right] \\ &= \left[(b+F(t_0)-F(t_1))/(\ell n(t_1)-\ell n(t_0)) \right] \\ c_U(n_{1.},n_{.1},n) &= \left[g^{-1}(a,n_{1.},n_{.1},t_0,t_1) \right] +1 \\ &= \left[(a+F(t_0)-F(t_1))/(\ell n(t_1)-\ell n(t_0)) \right] +1 \end{aligned} \tag{4.17}$$

where $F(t)=F(n_{1.},n_{.1},n,t)=\binom{n}{n_{.1}}\left\{\sum_j \binom{n_{1.}}{j}\binom{n-n_{1.}}{n_{.1}-j} t^j\right\}^{-1}$ and

M=[K] is the greatest integer less than or equal to K.

The sequential test region defined by these critical limits is used in the same manner as the regions discussed in Section 3.3. A numerical example of the above procedure follows. It is desired to test the hypothesis

$$H_0: \quad t=t_0=1$$
$$\text{versus } H_1: \quad t=t_1=9 \tag{4.18}$$

with desired error probabilities $\alpha=0.1$ and $\beta=0.25$. The test is truncated at trial 25. The critical limits for this test, which are shown in Tables 4.1a and 4.1b, were computed using the computer program for such tests which is listed in the Appendix. The limits for trials 1-10 are shown in Table 4.1a and Table 4.1b gives the limits for trial 25, the truncation point. The test procedure for the present case is exactly the same as explained

Table 4.1a

Critical Values for the

Sequential Test Example

T0= 1.000 ALPHA= 0.100

T1= 9.000 BETA= 0.250

TRIAL 5 N.1

N1.	0	1	2	3	4	5
0	-1; 1	-1, 1	-1, 1	-1, 1	-1, 1	:1, 1
1	-1; 1	-1, 2	0, 2	0, 2	0, 2	0, 2
2	-1; 1	0, 2	0, 3	0, 3	1, 3	1, 3
3	-1; 1	0, 2	0, 3	1, 4	2, 4	2, 4
4	-1; 1	0, 2	1, 3	2, 4	2, 5	3, 5
5	-1, 1	0, 2	1, 3	2, 4	3, 5	4, 6

TRIAL 6 N.1

N1.	0	1	2	3	4	5	6
0	-1; 1	-1, 1	-1, 1	-1, 1	-1, 1	:1, 1	-1, 1
1	-1; 1	-1, 2	0, 2	0, 2	0, 2	0, 2	0, 2
2	-1; 1	0, 2	0, 2	0, 3	1, 3	1, 3	1, 3
3	-1; 1	0, 2	0, 3	1, 3	1, 4	2, 4	2, 4
4	-1; 1	0, 2	1, 3	1, 4	2, 4	3, 5	3, 5
5	-1; 1	0, 2	1, 3	2, 4	3, 5	3, 6	4, 6
6	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7

TRIAL 7 N.1

N1.	0	1	2	3	4	5	6	7
0	-1; 1	-1, 1	-1, 1	-1, 1	-1, 1	:1, 1	-1, 1	-1, 1
1	-1; 1	-1, 2	-1, 2	0, 2	0, 2	0, 2	0, 2	0, 2
2	-1; 1	-1, 2	0, 2	0, 3	0, 3	1, 3	1, 3	1, 3
3	-1; 1	0, 2	0, 3	1, 3	1, 4	1, 4	2, 4	2, 4
4	-1; 1	0, 2	0, 3	1, 4	2, 4	2, 5	3, 5	3, 5
5	-1; 1	0, 2	1, 3	1, 4	2, 5	3, 5	3, 6	4, 6
6	-1; 1	0, 2	1, 3	2, 4	3, 5	3, 6	4, 7	5, 7
7	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8

TRIAL 8 N.1

N1.	0	1	2	3	4	5	6	7	8
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	:1, 1	-1, 1	-1, 1	-1, 1
1	-1; 1	-1, 2	-1, 2	0, 2	0, 2	0, 2	0, 2	0, 2	0, 2
2	-1; 1	-1, 2	0, 2	0, 3	0, 3	1, 3	1, 3	1, 3	1, 3
3	-1; 1	0, 2	0, 3	1, 3	1, 3	1, 4	2, 4	2, 4	2, 4
4	-1; 1	0, 2	0, 3	1, 3	2, 4	2, 4	2, 5	3, 5	3, 5
5	-1; 1	0, 2	1, 3	1, 4	2, 4	3, 5	3, 6	4, 6	4, 6
6	-1; 1	0, 2	1, 3	2, 4	2, 5	3, 6	4, 6	4, 7	5, 7
7	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	4, 7	5, 8	6, 8
8	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9

TRIAL 9 N.1

N1.	0	1	2	3	4	5	6	7	8	9
0	-1; 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1; 1	-1, 2	-1, 2	0, 2	0, 2	0, 2	0, 2	0, 2	0, 2	0, 2
2	-1; 1	-1, 2	0, 2	0, 3	0, 3	0, 3	1, 3	1, 3	1, 3	1, 3
3	-1; 1	0, 2	0, 3	0, 3	1, 3	1, 4	1, 4	2, 4	2, 4	2, 4
4	-1, 1	0, 2	0, 3	1, 3	1, 4	2, 4	2, 5	2, 5	3, 5	3, 5
5	-1; 1	0, 2	0, 3	1, 4	2, 4	2, 5	3, 5	3, 6	4, 6	4, 6
6	-1; 1	0, 2	1, 3	1, 4	2, 5	3, 5	3, 6	4, 7	5, 7	5, 7
7	-1, 1	0, 2	1, 3	2, 4	2, 5	3, 6	4, 7	5, 7	5, 8	6, 8
8	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	5, 8	6, 9	7, 9
9	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9	8,10

TRIAL 10 N.1

N1.	0	1	2	3	4	5	6	7	8	9	10
0	-1; 1	-1, 1	-1, 1	-1, 1	-1, 1	:1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1; 1	-1, 2	-1, 2	0, 2	0, 2	0, 2	0, 2	0, 2	0, 2	0, 2	0, 2
2	-1; 1	-1, 2	0, 2	0, 2	0, 3	0, 3	0, 3	1, 3	1, 3	1, 3	1, 3
3	-1; 1	-1, 2	0, 2	0, 3	1, 3	1, 4	1, 4	1, 4	2, 4	2, 4	2, 4
4	-1; 1	0, 2	0, 3	1, 3	1, 4	2, 4	2, 4	2, 5	2, 5	3, 5	3, 5
5	-1; 1	0, 2	0, 3	1, 4	2, 4	2, 5	3, 5	3, 6	3, 6	4, 6	4, 6
6	-1; 1	0, 2	0, 3	1, 4	2, 4	3, 5	3, 6	4, 6	4, 7	5, 7	5, 7
7	-1, 1	0, 2	1, 3	1, 4	2, 5	3, 6	4, 6	4, 7	5, 7	5, 8	6, 8
8	-1; 1	0, 2	1, 3	2, 4	2, 5	3, 6	4, 7	5, 7	6, 8	6, 9	7, 9
9	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	5, 8	6, 9	7,10	8,10
10	-1; 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9	8,10	9,11

TRIAL 1 N.1

N1.	0	1
0	-1; 1	-1, 1
1	-1; 1	0, 2

TRIAL 2 N.1

N1.	0	1	2
0	-1; 1	-1, 1	-1, 1
1	-1, 1	0, 2	0, 2
2	-1; 1	0, 2	1, 3

TRIAL 3 N.1

N1.	0	1	2	3
0	-1; 1	-1; 1	-1, 1	-1, 1
1	-1; 1	0, 2	0, 2	0, 2
2	-1; 1	0, 2	1, 3	1, 3
3	-1; 1	0, 2	1, 3	2, 4

TRIAL 4 N.1

N1.	0	1	2	3	4
0	-1; 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1; 1	-1, 2	0, 2	0, 2	0, 2
2	-1; 1	0, 2	0, 3	1, 3	1, 3
3	-1; 1	0, 2	1, 3	1, 4	2, 4
4	-1; 1	0, 2	1, 3	2, 4	3, 5

Table 4.1b

TRIAL N1. 25 N.1 T0= 1.000 T1= 9.000 ALPHA= 0.100 BETA= 0.250

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	19	16	17	18	19	20	21	22	23	24	25				
0	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1
	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1				
1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1
	0, 1	0, 1	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2				
2	0, 1	0, 1	0, 1	0, 1	0, 1	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2
	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	2, 3	2, 3	2, 3	2, 3	2, 3				
3	0, 1	0, 1	0, 1	0, 1	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	2, 3	2, 3	2, 3	2, 3	2, 3
	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3	3, 4	3, 4	3, 4	3, 4				
4	0, 1	0, 1	0, 1	1, 2	1, 2	1, 2	1, 2	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3	3, 4	3, 4
	3, 4	3, 4	3, 4	3, 4	3, 4	3, 4	3, 4	3, 4	4, 5	4, 5	4, 5				
5	0, 1	0, 1	1, 2	1, 2	1, 2	2, 3	2, 3	2, 3	2, 3	3, 4	3, 4	3, 4	3, 4	3, 4	3, 4
	4, 5	4, 5	4, 5	4, 5	4, 5	4, 5	4, 5	4, 5	4, 5	5, 6	5, 6				
6	0, 1	0, 1	1, 2	1, 2	1, 2	2, 3	2, 3	2, 3	3, 4	3, 4	3, 4	4, 5	4, 5	4, 5	4, 5
	4, 5	4, 5	5, 6	5, 6	5, 6	5, 6	5, 6	5, 6	5, 6	6, 7	6, 7				
7	0, 1	0, 1	1, 2	1, 2	2, 3	2, 3	2, 3	3, 4	3, 4	4, 5	4, 5	4, 5	4, 5	5, 6	5, 6
	5, 6	5, 6	6, 7	6, 7	6, 7	6, 7	6, 7	6, 7	6, 7	7, 8	7, 8				
8	0, 1	0, 1	1, 2	1, 2	2, 3	2, 3	3, 4	3, 4	4, 5	4, 5	4, 5	5, 6	5, 6	5, 6	6, 7
	6, 7	6, 7	6, 7	6, 7	7, 8	7, 8	7, 8	7, 8	7, 8	8, 9	8, 9				
9	0, 1	0, 1	1, 2	1, 2	2, 3	3, 4	3, 4	4, 5	4, 5	4, 5	5, 6	5, 6	6, 7	6, 7	6, 7
	6, 7	7, 8	7, 8	7, 8	7, 8	8, 9	8, 9	8, 9	8, 9	8, 9	9,10				
10	0, 1	0, 1	1, 2	2, 3	2, 3	3, 4	3, 4	4, 5	4, 5	5, 6	5, 6	6, 7	6, 7	6, 7	7, 8
	7, 8	7, 8	8, 9	8, 9	8, 9	9,10	9,10	9,10	9,10	9,10	10,11				
11	0, 1	0, 1	1, 2	2, 3	2, 3	3, 4	4, 5	4, 5	5, 6	5, 6	6, 7	6, 7	7, 8	7, 8	7, 8
	8, 9	8, 9	9,10	9,10	9,10	9,10	10,11	10,11	10,11	10,11	11,12				
12	0, 1	0, 1	1, 2	2, 3	2, 3	3, 4	4, 5	4, 5	5, 6	6, 7	6, 7	7, 8	7, 8	8, 9	8, 9
	8, 9	9,10	9,10	10,11	10,11	10,11	11,12	11,12	11,12	11,12	12,13				
13	0, 1	0, 1	1, 2	2, 3	3, 4	3, 4	4, 5	5, 6	5, 6	6, 7	6, 7	7, 8	8, 9	8, 9	9,10
	9,10	10,11	10,11	10,11	11,12	11,12	11,12	12,13	12,13	12,13	13,14				
14	0, 1	0, 1	1, 2	2, 3	3, 4	3, 4	4, 5	5, 6	6, 7	6, 7	7, 8	7, 8	8, 9	9,10	9,10
	10,11	10,11	11,12	11,12	12,13	12,13	12,13	13,14	13,14	13,14	14,15				
15	0, 1	0, 1	1, 2	2, 3	3, 4	4, 5	4, 5	5, 6	6, 7	6, 7	7, 8	8, 9	8, 9	9,10	10,11
	10,11	11,12	11,12	12,13	12,13	13,14	13,14	14,15	14,15	14,15	15,16				
16	0, 1	0, 1	1, 2	2, 3	3, 4	4, 5	4, 5	5, 6	6, 7	7, 8	7, 8	8, 9	9,10	10,11	10,11
	11,12	11,12	12,13	13,14	13,14	14,15	14,15	14,15	15,16	15,16	16,17				
17	0, 1	1, 2	1, 2	2, 3	3, 4	4, 5	5, 6	5, 6	6, 7	7, 8	8, 9	9,10	9,10	10,11	11,12
	11,12	12,13	13,14	13,14	14,15	14,15	15,16	15,16	16,17	16,17	17,18				
18	0, 1	1, 2	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	6, 7	7, 8	8, 9	9,10	10,11	10,11	11,12
	12,13	13,14	13,14	14,15	14,15	15,16	16,17	16,17	17,18	17,18	18,19				
19	0, 1	1, 2	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 8	7, 8	8, 9	9,10	10,11	11,12	12,13
	12,13	13,14	14,15	14,15	15,16	16,17	16,17	17,18	18,19	18,19	19,20				
20	0, 1	1, 2	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 8	8, 9	9,10	9,10	10,11	11,12	12,13
	13,14	14,15	14,15	15,16	16,17	17,18	17,18	18,19	19,20	19,20	20,21				
21	0, 1	1, 2	2, 3	2, 3	3, 4	4, 5	5, 6	6, 7	7, 8	8, 9	9,10	10,11	11,12	11,12	12,13
	13,14	14,15	15,16	16,17	16,17	17,18	18,19	19,20	19,20	20,21	21,22				
22	0, 1	1, 2	2, 3	3, 4	3, 4	4, 5	5, 6	6, 7	7, 8	8, 9	9,10	10,11	11,12	12,13	13,14
	14,15	14,15	15,16	16,17	17,18	18,19	19,20	19,20	20,21	21,22	22,23				
23	0, 1	1, 2	2, 3	3, 4	4, 5	4, 5	5, 6	6, 7	7, 8	8, 9	9,10	10,11	11,12	12,13	13,14
	14,15	15,16	16,17	17,18	18,19	19,20	19,20	20,21	21,22	22,23	23,24				
24	0, 1	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 8	8, 9	8, 9	9,10	10,11	11,12	12,13	13,14
	14,15	15,16	16,17	17,18	18,19	19,20	20,21	21,22	22,23	23,24	24,25				
25	0, 1	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 8	8, 9	9,10	10,11	11,12	12,13	13,14	14,15
	15,16	16,17	17,18	18,19	19,20	20,21	21,22	22,23	23,24	24,25	25,26				

at the end of Section 3.3, for the case when the marginal probabilities are known.

The construction of the sequential test regions for three decision test procedure is analogous to the development in Section 3.4. In this case, however, the three hypotheses are specified as

$$\begin{aligned} H_1: &\quad t=t_1<t_0 \\ \text{versus } H_0: &\quad t=t_0 \\ \text{versus } H_2: &\quad t=t_2>t_0 \end{aligned} \tag{4.19}$$

In addition, the desired α and β errors, α_1, β_1, α_2 and β_2, are specified as before. Two log likelihood ratios are then constructed as

$$\begin{aligned} \ell n(Ln_0/Ln_1)&=g(x,n_{1.},n_{.1},n,t_1,t_0) \\ \ell n(Ln_2/Ln_0)&=g(x,n_{1.},n_{.1},n,t_0,t_2) \end{aligned} \tag{4.20}$$

where $g(\cdot)$ is the same as (4.16) and the test procedure rules are the same as the ones shown in (3.25), giving rise to two sets of critical values, one each for the first and last pair of hypotheses in (4.19).

As a numerical example, consider the hypotheses

$$\begin{aligned} H_1: &\quad t=t_1=0.1111 \\ \text{versus } H_0: &\quad t=t_0=1.0 \\ \text{versus } H_2: &\quad t=t_2=9.0 \end{aligned} \tag{4.21}$$

and $\alpha_1=\alpha_2=0.1$ and $\beta_1=\beta_2=0.2$. α_1 and β_1 are again interchanged because $t_1<t_0$. The limits for the second pair of hypotheses are the same as those shown in Tables 4.1a and b. The critical limits for the first pair, up to trial 10, are shown in Table 4.2. The exact values for the properties of regions for both the two and three decision test procedures are found in the next section by using the direct method of sequential analysis.

4.4 EVALUATION OF THE SEQUENTIAL TEST REGIONS

This section describes the evaluation of the sequential test regions for a 2x2 contingency table when the marginal probabilities are unknown. The regions developed in the last section are evaluated here as a numerical example.

The method of finding the exact properties of the sequential test regions for the present model is essentially the same as the procedure used for the 2x2 contingency table with known marginal probabilities, treated in Section 3.5. This is due to the same underlying multinomial distribution. Because the marginal probabilities are unknown in the present case, they must be specified as part of the state of nature. This means that the OC function, the ASN function and the distribution of the DSN will be functions of three parameters. These can be specified in a number of ways. Because the sequential test is based on the cross product ratio, the state of nature is specified here by $p_{1.}, p_{.1}$ and the cross product ratio

$$t=\frac{p_{11}(1-p_{1.}-p_{.1}+p_{11})}{(p_{1.}-p_{11})(p_{.1}-p_{11})} \qquad (4.22)$$

Table 4.2

Critical Values for the

Sequential Test Example

TRIAL 5 N.1 TO= 0.111 ALPHA= 0.250
TI= 1.000 BETA= 0.100

N1.	0	1	2	3	4	5
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	0, 2
2	-1, 1	-1, 1	-1, 2	-1, 2	0, 2	1, 3
3	-1, 1	-1, 1	-1, 2	0, 3	1, 3	2, 4
4	-1, 1	-1, 2	0, 2	1, 3	2, 4	3, 5
5	-1, 1	0, 2	1, 3	2, 4	3, 5	4, 6

TRIAL 6 N.1

N1.	0	1	2	3	4	5	6
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	0, 2
2	-1, 1	-1, 1	-1, 1	-1, 2	0, 2	0, 2	1, 3
3	-1, 1	-1, 1	-1, 2	0, 2	0, 3	1, 3	2, 4
4	-1, 1	-1, 1	0, 2	0, 3	1, 3	2, 4	3, 5
5	-1, 1	-1, 2	0, 2	1, 3	2, 4	3, 5	4, 6
6	-1, 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7

TRIAL 7 N.1

N1.	0	1	2	3	4	5	6	7
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	0, 2
2	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	0, 2	0, 3	1, 3
3	-1, 1	-1, 1	-1, 2	-1, 2	0, 2	0, 3	1, 3	2, 4
4	-1, 1	-1, 1	-1, 2	0, 2	0, 3	1, 4	2, 4	3, 5
5	-1, 1	-1, 2	0, 2	0, 3	1, 4	2, 4	3, 5	4, 6
6	-1, 1	-1, 2	0, 3	1, 3	2, 4	3, 5	4, 6	5, 7
7	-1, 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8

TRIAL 8 N.1

N1.	0	1	2	3	4	5	6	7	8
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	0, 2
2	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	0, 2	0, 3	1, 3
3	-1, 1	-1, 1	-1, 1	-1, 2	0, 2	0, 2	0, 3	1, 3	2, 4
4	-1, 1	-1, 1	-1, 2	0, 2	0, 2	1, 3	1, 4	2, 4	3, 5
5	-1, 1	-1, 1	-1, 2	0, 2	1, 3	1, 4	2, 4	3, 5	4, 6
6	-1, 1	-1, 2	0, 2	0, 3	1, 4	2, 4	3, 5	4, 6	5, 7
7	-1, 1	-1, 2	0, 3	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8
8	-1, 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9

TRIAL 9 N.1

N1.	0	1	2	3	4	5	6	7	8	9
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	0, 2
2	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	-1, 2	0, 2	0, 3	1, 3
3	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	0, 2	0, 3	0, 3	1, 3	2, 4
4	-1, 1	-1, 1	-1, 2	-1, 2	0, 2	0, 3	1, 3	1, 4	2, 4	3, 5
5	-1, 1	-1, 1	-1, 2	0, 2	0, 3	1, 3	1, 4	2, 5	3, 5	4, 6
6	-1, 1	-1, 1	-1, 2	0, 3	1, 3	1, 4	2, 5	3, 5	4, 6	5, 7
7	-1, 1	-1, 2	0, 2	0, 3	1, 4	2, 5	3, 5	4, 6	5, 7	6, 8
8	-1, 1	-1, 2	0, 3	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9
9	-1, 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9	8,10

TRIAL 10 N.1

N1.	0	1	2	3	4	5	6	7	8	9	10
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	-1, 2	0, 2
2	-1, 1	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	-1, 2	0, 2	0, 3	1, 3	1, 3
3	-1, 1	-1, 1	-1, 1	-1, 2	-1, 2	-1, 2	0, 2	0, 3	1, 3	1, 4	2, 4
4	-1, 1	-1, 1	-1, 2	-1, 2	0, 2	0, 2	0, 3	1, 3	1, 4	2, 4	3, 5
5	-1, 1	-1, 1	-1, 2	0, 2	0, 2	0, 3	1, 3	1, 4	2, 5	3, 5	4, 6
6	-1, 1	-1, 1	-1, 2	0, 2	0, 3	1, 3	1, 4	2, 5	3, 6	4, 6	5, 7
7	-1, 1	-1, 2	0, 2	0, 3	1, 3	1, 4	2, 5	3, 6	4, 6	5, 7	6, 8
8	-1, 1	-1, 2	0, 2	1, 3	1, 4	2, 5	3, 6	4, 6	5, 7	6, 8	7, 9
9	-1, 1	-1, 2	0, 3	1, 4	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9	8,10
10	-1, 1	0, 2	1, 3	2, 4	3, 5	4, 6	5, 7	6, 8	7, 9	8,10	9,11

TRIAL 1 N.1

N1.	0	1
0	-1, 1	-1, 1
1	-1, 1	0, 2

TRIAL 2 N.1

N1.	0	1	2
0	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	0, 2
2	-1, 1	0, 2	1, 3

TRIAL 3 N.1

N1.	0	1	2	3
0	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	0, 2
2	-1, 1	-1, 1	0, 2	1, 3
3	-1, 1	0, 2	1, 3	2, 4

TRIAL 4 N.1

N1.	0	1	2	3	4
0	-1, 1	-1, 1	-1, 1	-1, 1	-1, 1
1	-1, 1	-1, 1	-1, 1	-1, 2	0, 2
2	-1, 1	-1, 1	-1, 2	0, 2	1, 3
3	-1, 1	-1, 2	0, 2	1, 3	2, 4
4	-1, 1	0, 2	1, 3	2, 4	3, 5

The procedure for determining the sequential test properties which is described in Section 3.6 is used here with one modification. As explained above, the state of nature will be specified by $(p_{1.}, p_{.1}, t)$. Because the state of nature must be specified in three dimensions, the test properties can be expressed in a graph only if two of the parameters are held constant. A contour plot can be used if one of the parameters is held constant. Tables of the important test properties, however, are given below.

For the two decision example given in Section 4.3, the truncated sequential test for

$$H_0: \ t=t_0=1 \qquad (4.23)$$
$$\text{versus } H_1: \ t=t_1=9$$

and $\alpha=0.1$ and $\beta=0.25$ is evaluated for $p_{1.}=.1(.1).5$, $p_{.1}=.1(.1)p_{1.}$ and $t=1(2)9$ (other values being unnecessary because of symmetry). The same values of the state of nature are used to evaluate the three decision example given in Section 4.3 to test the hypothesis

$$H_1: \ t=t_1=0.1111$$
$$\text{versus } H_0: \ t=t_0=1 \qquad (4.24)$$
$$\text{versus } H_2: \ t=t_2=9$$

with $\alpha_1=\alpha_2=0.1$ and $\beta_1=\beta_2=0.25$.

Both the OC and the ASN functions are given for these examples in Tables 4.3 and 4.4 for the two and three decision examples respectively.

Table 4.3
Test Properties for the
Two Decision Test Example

p_1	$p_{.1}$	p_{11}	t	λ	$P(H_0)$	$P(H_1)$	ASN	$P(c_{n_0-1})$
0.1	0.1	0.0100	1.000	1.00	0.87038	0.12961	21.24	0.65266
0.1	0.1	0.0220	3.000	2.20	0.69234	0.30765	22.00	0.71927
0.1	0.1	0.0300	5.000	3.00	0.58053	0.41947	22.16	0.72657
0.1	0.1	0.0350	7.000	3.50	0.50452	0.49548	22.12	0.71714
0.1	0.1	0.0390	9.000	3.90	0.44961	0.55038	22.01	0.70267
0.2	0.1	0.0200	1.000	1.00	0.86467	0.13532	18.34	0.46253
0.2	0.1	0.0390	3.000	1.95	0.64836	0.35163	19.97	0.57048
0.2	0.1	0.0500	5.000	2.50	0.51625	0.48374	20.32	0.58138
0.2	0.1	0.0570	7.000	2.85	0.43071	0.56928	20.34	0.56970
0.2	0.1	0.0620	9.000	3.10	0.37154	0.62845	20.25	0.55270
0.2	0.2	0.0400	1.000	1.00	0.87970	0.12029	14.50	0.23948
0.2	0.2	0.0730	3.000	1.83	0.61382	0.38617	16.86	0.35806
0.2	0.2	0.0890	5.000	2.22	0.44968	0.55031	17.21	0.35962
0.2	0.2	0.1000	7.000	2.50	0.34722	0.65278	17.10	0.33709
0.2	0.2	0.1080	9.000	2.70	0.27909	0.72090	16.85	0.31116
0.3	0.1	0.0300	1.000	1.00	0.85087	0.14912	16.77	0.39245
0.3	0.1	0.0530	3.000	1.77	0.61452	0.38547	19.17	0.53083
0.3	0.1	0.0640	5.000	2.13	0.47972	0.52027	19.81	0.55260
0.3	0.1	0.0710	7.000	2.37	0.39638	0.60362	20.01	0.54876
0.3	0.1	0.0750	9.000	2.50	0.34044	0.65956	20.07	0.53861
0.3	0.2	0.0600	1.000	1.00	0.88417	0.11583	12.78	0.17781
0.3	0.2	0.1000	3.000	1.67	0.60035	0.39964	15.59	0.29536
0.3	0.2	0.1190	5.000	1.98	0.42962	0.57037	16.04	0.29513
0.3	0.2	0.1310	7.000	2.18	0.32643	0.67356	15.96	0.27249
0.3	0.2	0.1390	9.000	2.32	0.25953	0.74047	15.75	0.24779
0.3	0.3	0.0900	1.000	1.00	0.89764	0.10235	10.92	0.10730
0.3	0.3	0.1410	3.000	1.57	0.59686	0.40313	13.82	0.20483
0.3	0.3	0.1640	5.000	1.82	0.41426	0.58573	14.17	0.19828
0.3	0.3	0.1790	7.000	1.99	0.30629	0.69371	13.99	0.17376
0.3	0.3	0.1890	9.000	2.10	0.23793	0.76206	13.68	0.14952
0.4	0.1	0.0400	1.000	1.00	0.81847	0.18152	16.23	0.38027
0.4	0.1	0.0640	3.000	1.60	0.56767	0.43233	19.24	0.54997
0.4	0.1	0.0740	5.000	1.85	0.43663	0.56336	20.16	0.58911
0.4	0.1	0.0798	7.000	1.97	0.35909	0.64090	20.55	0.59899
0.4	0.1	0.0830	9.000	2.07	0.30814	0.69185	20.76	0.60011
0.4	0.2	0.0800	1.000	1.00	0.87750	0.12250	12.27	0.17214
0.4	0.2	0.1230	3.000	1.54	0.58243	0.41757	15.49	0.29740
0.4	0.2	0.1410	5.000	1.76	0.41278	0.58721	16.14	0.30122
0.4	0.2	0.1520	7.000	1.90	0.31300	0.68700	16.22	0.28274
0.4	0.2	0.1590	9.000	1.99	0.24929	0.75071	16.14	0.26198
0.4	0.3	0.1200	1.000	1.00	0.89897	0.10102	10.26	0.09047
0.4	0.3	0.1760	3.000	1.47	0.58981	0.41019	13.31	0.18280
0.4	0.3	0.2000	5.000	1.67	0.40547	0.59453	13.70	0.17589
0.4	0.3	0.2150	7.000	1.79	0.29835	0.70164	13.56	0.15272
0.4	0.3	0.2250	9.000	1.87	0.23135	0.76864	13.29	0.13038
0.4	0.4	0.1600	1.000	1.00	0.90448	0.09551	9.38	0.08275
0.4	0.4	0.2230	3.000	1.39	0.59000	0.40999	12.34	0.14090
0.4	0.4	0.2500	5.000	1.56	0.40133	0.59866	12.65	0.13228
0.4	0.4	0.2670	7.000	1.67	0.29274	0.70725	12.43	0.11019
0.4	0.4	0.2780	9.000	1.74	0.22555	0.77445	12.10	0.08990
0.5	0.1	0.0500	1.000	1.00	0.77770	0.22229	16.52	0.41334
0.5	0.1	0.0730	3.000	1.46	0.51859	0.48140	19.91	0.61142
0.5	0.1	0.0810	5.000	1.62	0.39573	0.60427	20.99	0.66701
0.5	0.1	0.0860	7.000	1.72	0.32569	0.67430	21.50	0.68962
0.5	0.1	0.0880	9.000	1.76	0.28067	0.71932	21.80	0.70080
0.5	0.2	0.1000	1.000	1.00	0.86100	0.13899	12.58	0.19941
0.5	0.2	0.1420	3.000	1.42	0.55815	0.44185	16.23	0.34498
0.5	0.2	0.1580	5.000	1.58	0.39481	0.60519	17.11	0.35835
0.5	0.2	0.1670	7.000	1.67	0.30101	0.69899	17.39	0.34747
0.5	0.2	0.1720	9.000	1.72	0.24162	0.75838	17.46	0.33281
0.5	0.3	0.1500	1.000	1.00	0.89125	0.10875	10.41	0.10179
0.5	0.3	0.2060	3.000	1.37	0.57388	0.42612	13.73	0.20200
0.5	0.3	0.2280	5.000	1.52	0.39294	0.60705	14.28	0.19706
0.5	0.3	0.2410	7.000	1.61	0.28975	0.71024	14.26	0.17534
0.5	0.3	0.2500	9.000	1.67	0.22568	0.77432	14.11	0.15405
0.5	0.4	0.2000	1.000	1.00	0.90186	0.09814	9.29	0.06206
0.5	0.4	0.2640	3.000	1.32	0.58262	0.41738	12.33	0.14036
0.5	0.4	0.2910	5.000	1.45	0.39484	0.60515	12.69	0.13189
0.5	0.4	0.3070	7.000	1.53	0.28786	0.71213	12.51	0.11025
0.5	0.4	0.3180	9.000	1.59	0.22199	0.77801	12.23	0.09045
0.5	0.5	0.2500	1.000	1.00	0.90482	0.09518	8.95	0.05139
0.5	0.5	0.3170	3.000	1.27	0.58635	0.41365	11.89	0.12325
0.5	0.5	0.3450	5.000	1.38	0.39687	0.60312	12.18	0.11468
0.5	0.5	0.3630	7.000	1.45	0.28877	0.71122	11.95	0.09390
0.5	0.5	0.3750	9.000	1.50	0.22230	0.77769	11.62	0.07518

Table 4.4
Test Properties for the
Three Decision Test Sample

P_1	$P_{.1}$	P_{11}	t	A	$P(H_0)$	$P(H_1)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.1	0.1	0.0100	1.000	1.00	0.10915	0.75646	0.13438	24.87	0.97853
0.1	0.1	0.0220	3.000	2.20	0.04654	0.63302	0.31844	24.40	0.92311
0.1	0.1	0.0300	5.000	3.00	0.02210	0.54410	0.43279	23.94	0.87145
0.1	0.1	0.0350	7.000	3.50	0.01536	0.47492	0.50971	23.52	0.82759
0.1	0.1	0.0390	9.000	3.90	0.01019	0.42905	0.56476	23.17	0.79073
0.2	0.1	0.0200	1.000	1.00	0.24259	0.61447	0.14294	24.67	0.94905
0.2	0.1	0.0390	3.000	1.95	0.08927	0.53969	0.37103	23.79	0.84873
0.2	0.1	0.0500	5.000	2.50	0.04622	0.44658	0.50719	23.03	0.77136
0.2	0.1	0.0570	7.000	2.85	0.02812	0.37835	0.59352	22.42	0.71115
0.2	0.1	0.0620	9.000	3.10	0.01884	0.32896	0.65220	21.93	0.66380
0.2	0.2	0.0400	1.000	1.00	0.27760	0.58697	0.13543	23.92	0.82911
0.2	0.2	0.0730	3.000	1.83	0.08037	0.48984	0.42978	22.29	0.68006
0.2	0.2	0.0890	5.000	2.22	0.03706	0.35963	0.60330	20.98	0.57454
0.2	0.2	0.1000	7.000	2.50	0.02080	0.27209	0.70710	19.95	0.49443
0.2	0.2	0.1080	9.000	2.70	0.01304	0.21338	0.77357	19.13	0.43278
0.3	0.1	0.0300	1.000	1.00	0.27437	0.56770	0.15791	24.47	0.92226
0.3	0.1	0.0530	3.000	1.77	0.09393	0.49628	0.40778	23.57	0.82492
0.3	0.1	0.0640	5.000	2.13	0.04927	0.40459	0.54613	22.88	0.75237
0.3	0.1	0.0710	7.000	2.37	0.03098	0.33950	0.62952	22.36	0.69837
0.3	0.1	0.0750	9.000	2.50	0.02157	0.29408	0.68434	21.96	0.65736
0.3	0.2	0.0600	1.000	1.00	0.22263	0.64452	0.13275	22.86	0.70252
0.3	0.2	0.1000	3.000	1.67	0.05340	0.49519	0.45140	21.24	0.57629
0.3	0.2	0.1190	5.000	1.98	0.02370	0.34340	0.63290	19.95	0.48466
0.3	0.2	0.1310	7.000	2.18	0.01331	0.24964	0.73704	18.94	0.41269
0.3	0.2	0.1390	9.000	2.32	0.00847	0.19005	0.80147	18.13	0.35706
0.3	0.3	0.0900	1.000	1.00	0.16867	0.71167	0.11965	20.80	0.47732
0.3	0.3	0.1410	3.000	1.57	0.02743	0.50742	0.46515	19.32	0.40188
0.3	0.3	0.1640	5.000	1.82	0.01025	0.32686	0.66288	18.01	0.33498
0.3	0.3	0.1790	7.000	1.99	0.00522	0.22206	0.77271	16.92	0.27579
0.3	0.3	0.1890	9.000	2.10	0.00312	0.15874	0.83814	16.04	0.22878
0.4	0.1	0.0400	1.000	1.00	0.26098	0.54963	0.18938	24.27	0.89542
0.4	0.1	0.0640	3.000	1.60	0.08934	0.45916	0.45149	23.68	0.83094
0.4	0.1	0.0740	5.000	1.85	0.05000	0.36034	0.58465	23.23	0.78210
0.4	0.1	0.0790	7.000	1.97	0.03369	0.30459	0.66171	22.90	0.74580
0.4	0.1	0.0830	9.000	2.07	0.02504	0.26366	0.71130	22.66	0.71848
0.4	0.2	0.0800	1.000	1.00	0.17914	0.68262	0.13823	21.88	0.59230
0.4	0.2	0.1230	3.000	1.54	0.03595	0.49708	0.46696	20.82	0.52675
0.4	0.2	0.1410	5.000	1.76	0.01558	0.33615	0.64627	19.86	0.46272
0.4	0.2	0.1520	7.000	1.90	0.00891	0.24455	0.74653	19.08	0.40670
0.4	0.2	0.1590	9.000	1.99	0.00587	0.18644	0.80768	18.46	0.36177
0.4	0.3	0.1200	1.000	1.00	0.14105	0.74238	0.11657	19.21	0.33794
0.4	0.3	0.1760	3.000	1.47	0.01739	0.51216	0.47044	18.30	0.32016
0.4	0.3	0.2000	5.000	1.67	0.00565	0.32414	0.67020	17.25	0.27751
0.4	0.3	0.2150	7.000	1.79	0.00268	0.21795	0.77936	16.31	0.23149
0.4	0.3	0.2250	9.000	1.87	0.00154	0.15484	0.84361	15.52	0.19311
0.4	0.4	0.1600	1.000	1.00	0.12249	0.76622	0.10929	17.39	0.21097
0.4	0.4	0.2230	3.000	1.39	0.01213	0.51724	0.47062	16.85	0.22627
0.4	0.4	0.2500	5.000	1.56	0.00337	0.31970	0.67692	15.90	0.19948
0.4	0.4	0.2670	7.000	1.67	0.00142	0.21004	0.78854	14.97	0.16310
0.4	0.4	0.2780	9.000	1.74	0.00074	0.14593	0.85332	14.18	0.13185
0.5	0.1	0.0500	1.000	1.00	0.22761	0.54484	0.22755	24.18	0.88063
0.5	0.1	0.0730	3.000	1.46	0.07725	0.42940	0.49334	24.00	0.86470
0.5	0.1	0.0810	5.000	1.62	0.04500	0.33816	0.61683	23.79	0.84154
0.5	0.1	0.0860	7.000	1.72	0.03150	0.28238	0.68610	23.64	0.82335
0.5	0.1	0.0880	9.000	1.76	0.02416	0.24570	0.73013	23.53	0.80928
0.5	0.2	0.1000	1.000	1.00	0.15318	0.69378	0.15303	21.47	0.54676
0.5	0.2	0.1420	3.000	1.42	0.02744	0.48793	0.48462	21.13	0.54090
0.5	0.2	0.1580	5.000	1.58	0.01187	0.33325	0.65487	20.56	0.50222
0.5	0.2	0.1670	7.000	1.67	0.00701	0.24477	0.74821	20.06	0.46188
0.5	0.2	0.1720	9.000	1.72	0.00481	0.19019	0.80499	19.65	0.42778
0.5	0.3	0.1500	1.000	1.00	0.12408	0.75192	0.12399	18.63	0.29293
0.5	0.3	0.2060	3.000	1.37	0.01377	0.50326	0.48296	18.34	0.31422
0.5	0.3	0.2280	5.000	1.52	0.00427	0.31874	0.67698	17.60	0.28559
0.5	0.3	0.2410	7.000	1.61	0.00197	0.21665	0.78137	16.87	0.24736
0.5	0.3	0.2500	9.000	1.67	0.00113	0.15624	0.84263	16.25	0.21384
0.5	0.4	0.2000	1.000	1.00	0.11167	0.77668	0.11164	16.77	0.17520
0.5	0.4	0.2640	3.000	1.32	0.01054	0.51318	0.47627	16.57	0.20903
0.5	0.4	0.2910	5.000	1.45	0.00290	0.31635	0.68074	15.78	0.18878
0.5	0.4	0.3070	7.000	1.53	0.00121	0.20832	0.79046	14.95	0.15629
0.5	0.4	0.3180	9.000	1.59	0.00062	0.14542	0.85395	14.23	0.12762
0.5	0.5	0.2500	1.000	1.00	0.10782	0.78436	0.10781	16.14	0.14311
0.5	0.5	0.3170	3.000	1.27	0.00990	0.51787	0.47223	15.98	0.17989
0.5	0.5	0.3450	5.000	1.38	0.00269	0.31772	0.67958	15.18	0.16255
0.5	0.5	0.3630	7.000	1.45	0.00110	0.20782	0.79107	14.32	0.13242
0.5	0.5	0.3750	9.000	1.50	0.00056	0.14399	0.85544	13.58	0.10590

From Table 4.3, it can be seen that the values of the power function (i.e., $P(H_1)$), for different states of nature where t=1, approach or achieve the desired error probability, α, for most values of the nuisance parameters, $p_{1.}$ and $p_{.1}$. Also, the β error probabilities (i.e., the probability of accepting H_0 when t=9) vary with the nuisance parameters, but approach or achieve the desired value (β=0.25) in most cases. The test is shown to be more powerful if one of the nuisance parameters has values close to 0.5 as opposed to extreme values close to 0 or 1. The ASN function for this test varies between 8.95 and 21.80.

The results for the three decision test are similar. Here, however, the power has been reduced somewhat and the ASN function is generally larger. This is due to additional hypotheses under consideration. The test properties are still generally acceptable for most values of the nuisance parameters and modification of the test region, as explained in Chapter 7, will enable one to adjust the test properties to be within the desired limits.

Table 4.5 shows the test properties for the three decision numerical example, with the region modified (as explained above) to favor H_0 at the truncation trial n_0, reducing the α error probability; this has caused a corresponding loss of power for the test. This modification was made to facilitate comparisons with the power of the fixed size test as given by Harkness (1959). Table 4.6 shows, for a range of parameter values,

$$P(H_a;p_{1.},p_{.1},t)=P(H_1;p_{1.},p_{.1},t)+P(H_2;p_{1.},p_{.1},t), \quad (4.25)$$

Table 4.5
Test Properties for the
Three Decision Test Example,
(Favoring H_0)

p_1	$p_{.1}$	p_{11}	t	λ	$P(H_1)$	$P(H_0)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.1	0.1	0.0100	1.000	1.00	0.00012	0.98537	0.01451	24.87	0.97853
0.1	0.1	0.0220	3.000	2.20	0.00003	0.92884	0.07112	24.40	0.92311
0.1	0.1	0.0300	5.000	3.00	0.00002	0.87366	0.12630	23.94	0.87149
0.1	0.1	0.0350	7.000	3.50	0.00001	0.82675	0.17323	23.52	0.82759
0.1	0.1	0.0390	9.000	3.90	0.00000	0.78744	0.21255	23.17	0.79073
0.2	0.1	0.0200	1.000	1.00	0.00158	0.96893	0.02948	24.67	0.94505
0.2	0.1	0.0390	3.000	1.99	0.00029	0.86943	0.13028	23.79	0.84873
0.2	0.1	0.0500	5.000	2.50	0.00010	0.78391	0.21598	23.03	0.77136
0.2	0.1	0.0570	7.000	2.85	0.00005	0.71795	0.28199	22.42	0.71115
0.2	0.1	0.0620	9.000	3.10	0.00003	0.66672	0.33324	21.93	0.66380
0.2	0.2	0.0400	1.000	1.00	0.01231	0.93662	0.05106	23.92	0.87911
0.2	0.2	0.0730	3.000	1.83	0.00151	0.77259	0.22589	22.29	0.68006
0.2	0.2	0.0890	5.000	2.22	0.00044	0.63389	0.36567	20.98	0.57454
0.2	0.2	0.1000	7.000	2.50	0.00018	0.53286	0.46714	19.95	0.49443
0.2	0.2	0.1080	9.000	2.70	0.00009	0.45784	0.54205	19.13	0.43278
0.3	0.1	0.0300	1.000	1.00	0.00625	0.96000	0.03374	24.47	0.92226
0.3	0.1	0.0530	3.000	1.77	0.00090	0.85587	0.14322	23.57	0.82492
0.3	0.1	0.0640	5.000	2.13	0.00030	0.77136	0.22834	22.88	0.75237
0.3	0.1	0.0710	7.000	2.37	0.00014	0.71004	0.28982	22.36	0.69837
0.3	0.1	0.0750	9.000	2.50	0.00008	0.66454	0.33538	21.96	0.65736
0.3	0.2	0.0600	1.000	1.00	0.03027	0.91040	0.05933	22.86	0.70252
0.3	0.2	0.1000	3.000	1.67	0.00326	0.73265	0.26409	21.24	0.57629
0.3	0.2	0.1190	5.000	1.98	0.00091	0.58038	0.41870	19.95	0.48466
0.3	0.2	0.1310	7.000	2.18	0.00038	0.47409	0.52553	18.94	0.41269
0.3	0.2	0.1390	9.000	2.32	0.00019	0.39842	0.60138	18.13	0.35706
0.3	0.3	0.0900	1.000	1.00	0.05469	0.87483	0.07047	20.80	0.47732
0.3	0.3	0.1410	3.000	1.57	0.00541	0.67457	0.32001	19.32	0.40188
0.3	0.3	0.1640	5.000	1.82	0.00147	0.49788	0.50064	18.01	0.33498
0.3	0.3	0.1790	7.000	1.99	0.00059	0.37977	0.61963	16.92	0.27579
0.3	0.3	0.1890	9.000	2.10	0.00030	0.29925	0.70044	16.04	0.22878
0.4	0.1	0.0400	1.000	1.00	0.01414	0.95587	0.02999	24.27	0.89542
0.4	0.1	0.0640	3.000	1.60	0.00187	0.87417	0.12395	23.68	0.83094
0.4	0.1	0.0740	5.000	1.85	0.00061	0.80809	0.19129	23.23	0.78210
0.4	0.1	0.0790	7.000	1.97	0.00029	0.76231	0.23740	22.90	0.74580
0.4	0.1	0.0830	9.000	2.07	0.00016	0.72545	0.27038	22.66	0.71848
0.4	0.2	0.0800	1.000	1.00	0.04639	0.89327	0.06034	21.88	0.59230
0.4	0.2	0.1230	3.000	1.54	0.00494	0.72768	0.26737	20.82	0.52675
0.4	0.2	0.1410	5.000	1.76	0.00144	0.58246	0.41609	19.86	0.46272
0.4	0.2	0.1520	7.000	1.90	0.00062	0.48368	0.51570	19.08	0.40670
0.4	0.2	0.1590	9.000	1.99	0.00033	0.41453	0.58514	18.46	0.36177
0.4	0.3	0.1200	1.000	1.00	0.06861	0.85693	0.07446	19.21	0.33794
0.4	0.3	0.1760	3.000	1.47	0.00684	0.65433	0.33882	18.30	0.32016
0.4	0.3	0.2000	5.000	1.67	0.00191	0.47463	0.52345	17.25	0.27751
0.4	0.3	0.2150	7.000	1.79	0.00079	0.35752	0.64168	16.31	0.23149
0.4	0.3	0.2250	9.000	1.87	0.00041	0.27928	0.72030	15.52	0.19311
0.4	0.4	0.1600	1.000	1.00	0.07813	0.84202	0.07986	17.39	0.21097
0.4	0.4	0.2230	3.000	1.39	0.00775	0.62521	0.36703	16.85	0.22627
0.4	0.4	0.2500	5.000	1.56	0.00218	0.43456	0.56325	15.90	0.19948
0.4	0.4	0.2670	7.000	1.67	0.00091	0.31363	0.68546	14.97	0.16310
0.4	0.4	0.2780	9.000	1.74	0.00046	0.23502	0.76452	14.18	0.13185
0.5	0.1	0.0500	1.000	1.00	0.02305	0.95427	0.02267	24.18	0.88063
0.5	0.1	0.0730	3.000	1.46	0.00303	0.90803	0.08893	24.00	0.86400
0.5	0.1	0.0810	5.000	1.62	0.00102	0.86680	0.13217	23.79	0.84155
0.5	0.1	0.0860	7.000	1.72	0.00049	0.83930	0.16020	23.64	0.82333
0.5	0.1	0.0880	9.000	1.76	0.00028	0.82012	0.17958	23.53	0.80928
0.5	0.2	0.1000	1.000	1.00	0.05662	0.88722	0.05615	21.47	0.54676
0.5	0.2	0.1420	3.000	1.42	0.00632	0.74913	0.24453	21.13	0.54090
0.5	0.2	0.1580	5.000	1.58	0.00196	0.62423	0.37380	20.56	0.50222
0.5	0.2	0.1670	7.000	1.67	0.00089	0.54050	0.45861	20.06	0.46188
0.5	0.2	0.1720	9.000	1.72	0.00049	0.48623	0.51728	19.65	0.42778
0.5	0.3	0.1500	1.000	1.00	0.07423	0.85174	0.07403	18.63	0.29293
0.5	0.3	0.2060	3.000	1.37	0.00774	0.65862	0.33363	18.34	0.31422
0.5	0.3	0.2280	5.000	1.52	0.00229	0.48792	0.50979	17.60	0.28559
0.5	0.3	0.2410	7.000	1.61	0.00100	0.37762	0.62138	16.87	0.24736
0.5	0.3	0.2500	9.000	1.67	0.00053	0.30392	0.69554	16.25	0.21384
0.5	0.4	0.2000	1.000	1.00	0.08092	0.83621	0.08287	16.77	0.17520
0.5	0.4	0.2640	3.000	1.32	0.00820	0.62081	0.37098	16.97	0.28903
0.5	0.4	0.2910	5.000	1.45	0.00237	0.43142	0.56620	15.78	0.18878
0.5	0.4	0.3070	7.000	1.53	0.00101	0.31230	0.68669	14.95	0.15625
0.5	0.4	0.3180	9.000	1.59	0.00053	0.23521	0.76425	14.23	0.12762
0.5	0.5	0.2500	1.000	1.00	0.08255	0.83492	0.08253	16.14	0.14311
0.5	0.5	0.3170	3.000	1.27	0.00831	0.61122	0.38046	15.98	0.17989
0.5	0.5	0.3450	5.000	1.38	0.00239	0.41726	0.58035	15.18	0.16255
0.5	0.5	0.3630	7.000	1.45	0.00101	0.29629	0.70269	14.32	0.13242
0.5	0.5	0.3750	9.000	1.50	0.00052	0.21882	0.78066	13.58	0.10590

Table 4.6
Comparison with Fixed Size Tests

			Fixed Size Tests		Sequential Test	
$p_{1.}$	$p_{.1}$	λ^*	$P_{20}(H_1)$	$P_{30}(H_1)$	$P_s(H_1)$	ASN
.1	.1	1.00	.050	.050	.015	24.87
.1	.1	3.90	.109		.213	23.17
.2	.1	1.00	.050	.050	.031	24.67
.2	.1	3.10	.153		.333	21.93
.2	.2	1.00	.050	.050	.063	23.92
.2	.2	2.70	.283		.542	19.13
.3	.1	1.00	.050	.050	.040	24.47
.3	.1	2.50	.157	.269	.336	21.96
.3	.2	1.00	.050	.050	.090	22.86
.3	.2	2.32	.340	.534	.602	18.13
.3	.3	1.00	.050	.050	.125	20.80
.3	.3	2.10	.448	.658	.701	16.04
.4	.1	1.00	.050	.050	.044	24.27
.4	.1	2.07	.139	.242	.542	19.13
.4	.2	1.00	.050	.050	.107	21.88
.4	.2	1.99	.333	.532	.590	18.46
.4	.3	1.00	.050	.050	.143	19.21
.4	.3	1.87	.474	.684	.721	15.52
.4	.4	1.00	.050	.050	.158	17.34
.4	.4	1.74	.546	.754	.765	14.18
.5	.1	1.00	.050	.050	.046	24.18
.5	.1	1.76	.115	.195	.180	23.53
.5	.2	1.00	.050	.050	.113	21.47
.5	.2	1.72	.282	.472	.518	23.64
.5	.3	1.00	.050	.050	.159	18.63
.5	.3	1.67	.453	.666	.696	16.25
.5	.4	1.00	.050	.050	.162	16.77
.5	.4	1.59	.520	.750	.765	14.23
.5	.5	1.00	.050	.050	.164	16.14
.5	.5	1.50	.577	.780	.782	13.58

*$\lambda = p_{11}/((p_{1.})(p_{.1}))$

(which is the power function) for the sequential test ($P_s(\cdot)$) whose properties are shown fully in Table 4.5 and for the fixed size sample tests ($P_n(\cdot)$, as given by Harkness (1959). The fixed size test properties are given for sample sizes of 20 and 30. The missing values in the table were not provided by Harkness. The ASN function for the sequential test is also shown in Table 4.6.

It can be seen from Table 4.6, using the value of the ASN function to decide which fixed size procedure to compare with for different points in the parameter space, that the sequential procedure has considerable advantage. Where the error probabilities are comparable, the ASN function is considerably smaller than the fixed size test necessary to obtain the same power.

CHAPTER 5

A NEW SEQUENTIAL TEST FOR THE EQUALITY OF TWO UNKNOWN BINOMIAL PROPORTIONS

5.0 INTRODUCTION

This chapter presents a new sequential test for the equality of two unknown binomial proportions. Several other such tests have been suggested in the past; a brief review of the relevant literature is contained in the first section, followed by a description of the underlying probability model. Section 5.2 develops the theory for the sequential test and the following section describes the evaluation of the resulting sequential test regions. Both two and three decision test procedures are considered. The last section gives further numerical examples and compares the tests with some other similar tests, both fixed size and sequential.

5.1 TESTS WHICH COMPARE TWO UNKNOWN BINOMIAL PROPORTIONS

One of the most common statistical problems arising in practice is the comparison of two unknown binomial proportions. It occurs, for example, when comparing two drug treatments, two production processes, or two teaching methods. The underlying probability model of this situation is depicted in Figure 5.1, where p_1 is the probability that a member of population 1 selected at random will have attribute D; p_2 is the same probability for population 2.

	D	$\bar{D}$
Population 1	p_1	$1-p_1$
Population 2	p_2	$1-p_2$

Figure 5.1 Probability Model for the Two-Sample Binomial Problem

A sample arising in such a situation with n_1 observations from population 1 and n_2 observations from population 2 is represented tabularly in Figure 5.2.

	D	$\bar{D}$	
Population 1	x	n_1-x	n_1
Population 2	y	n_2-y	n_2

Figure 5.2 Observed Data from a Two-Sample Binomial Experiment

The probability of observing the sample in Figure 5.2 is the joint distribution of two independent binomial distributions. That is,

$$P(x,y;n_{1.},n_{.1},p_1,p_2)=\binom{n_1}{x} p_1(1-p_1)^{n_1-x} \binom{n_2}{y} p_2(1-p_2)^{n_2-y} \quad (5.1)$$

The hypothesis usually being tested in this situation is

$$H_0: \quad p_1 = p_2$$
$$\text{versus } H_1: \quad p_1 \neq p_2 \tag{5.2}$$

and the test can be either a one-sided (two decision) or a two-sided (three decision) procedure. Tests of such hypotheses are treated in detail, for example, by Fleiss (1973)

It can be shown that the type of tests considered here are asymptotically most powerful if equal sample sizes are taken from each population (Lehmann, 1959). For small samples, the amount of information obtained is dependent on the sample outcome. For this reason, and for simplicity, although the results presented here are perfectly general, it will be assumed that n_1 and n_2, shown in Figure 5.2, are equal; sequential tests for this special case are developed here.

For large samples, the central limit theorem allows the use of the normal approximation for this test; this is equivalent to the χ^2 test with one degree of freedom and was first used by Karl Pearson (1900). For small samples, Fisher's exact test, as described in Section 4.2, is appropriate.

Because one margin is controlled by the experimenter, Fisher's exact test is conditioned on the one remaining random margin. As shown in Section 4.2, the hypergeometric is again the null distribution. This test has also been criticized because it limits the reference set of possible outcomes; however, it is now generally accepted as correct. The power of this test has been evaluated, for example, by Bennett and Hsu (1960) and Harkness (1959).

It should be noted that the probability model in (5.1) is not correct if there is "pairing" within observations. This occurs if, at each trial, observations are not procured at random, but rather chosen in pairs from different strata (which will affect the frequency of a given response). That is, each pair is matched with respect to some characteristics (e.g., by age when testing the value of two new drugs). Such "pairing" is often used to reduce the variability between the individual observations and can result in a more powerful test. The extreme case of such pairing occurs in drug testing, for example, when a patient receives both treatments being tested at different times. Thus, there is some correlation between the treatments. McNemar (1949) and Cochran (1950) treat such tests. A comprehensive review of this subject is given by Fleiss (1973). The sequential tests presented here assume that the two treatments are assigned to subjects at random or that one observation is taken at random from each population at each trial in order to compare the two unknown binomial proportions; that is, there is no "pairing" of the observations.

The odds ratio

$$t = \frac{p_1(1-p_2)}{p_2(1-p_1)}, \qquad (5.3)$$

on which these tests are based, is analogous to the cross product ratio discussed in Section 4.1. If $t=1$, p_1 and p_2 are equal. If $t>1$, p_1 is greater than p_2 and if $t<1$, p_1 is less than p_2. As explained in Section 4.1, the odds ratio is the most appropriate

method of comparing two proportions over a wide range of parameter values. Table 4.1 shows the odds ratio as a function of p_1 and p_2. This table can be used to aid one in choosing the proper values of the odds ratio to use in a given test situation.

5.2 CONSTRUCTION OF THE SEQUENTIAL TEST REGIONS FOR TWO AND THREE DECISION TEST PROCEDURES

In this section, the literature concerning sequential tests for comparing two binomial proportions is briefly reviewed. Following this, it is shown how the results of the last chapter can be modified to solve such problems sequentially and with a sufficient statistic.

There have been many suggested sequential procedures for comparing two binomial proportions, as explained above. The important ones are mentioned here; a more thorough review is given, for example, by Öksoy (1972). Wald (1947) suggests a procedure which ignores ties when they occur and uses the test statistic D=x-y, where x and y are the number of observed successes for populations 1 and 2 respectively. The test then reduces to a test of a single binomial proportion. Wald comments that because the statistic D is not sufficient, this procedure is not in general optimal. It can be shown, however, by using the two sample sequential procedure of Girshick (1946), that D is sufficient for testing the special case

$$\begin{aligned} H_0&: \quad t=t_0\neq 1 \\ \text{versus } H_1&: \quad t=1/t_0 \end{aligned} \tag{5.4}$$

where t is the hypothesized odds ratio shown in (5.3).

Ghosh (1970) reviews the theory of this test. He also treats in detail the concept of Fraser sufficiency (Fraser, 1956) which can be used to treat certain problems with nuisance parameters. For the important cases when one must test hypotheses different than (5.4) (e.g., for the equality of p_1 and p_2, implying t=1), D=x-y is no longer sufficient. Wald, however, points out that for large samples there is little loss of efficiency. The test presented here uses a sufficient statistic and is valid for small samples.

Tests similar to the above have been used by several authors to test the null hypothesis $p_1=p_2$; these include Bross (1952), Armitage (1960), Choi (1968), Öksoy (1972), and Elfring and Schultz (1973a). Except for being truncated, most of these plans have regions similar to those proposed by Wald (1947). In practice, such tests are almost always truncated at some trial n_0 in order to eliminate the possibility of large sample sizes. This is especially true for certain applications such as medical trials. In the papers of Öksoy (1972) and Elfring and Schultz (1973a), the sequential test properties are found exactly by using the direct method of sequential analysis. Also, their tests are truncated at a fixed trial, rather than at a fixed number of untied pairs, as is the case with the tests of Armitage and Wald, for example. These tests can be made quite efficient if one knows in advance the approximate values of p_1 and p_2. This is done by a trial and error procedure, comparing the

test properties for alternate tests plans, as explained by Öksoy (1972). Hall (1965) suggests a sequential test which is conditional on the observed ancillary statistic at each trial.

If the sequential test is based on a sufficient statistic, a powerful test can be found over a much wider ranger of parameter values. The sequential tests presented here, a special case of the tests given in Chapter 4, are based on such a statistic.

It will be convenient to use the notation of Figure 3.2 with the following modification. It is assumed that the right-hand margin is controlled such that equal sample sizes are taken from each population. Thus $n_{1.}=n/2$ for all n. The quantity $n_{.1}$ is still a random variable and equal to the total number of successes found in both populations (and $n_{.1}-x$ is the number of successes found in population 2). The joint distribution of the sample $(x,n_{.1}-x)$ at trial $n_{1.}$ is

$$P(x,n_{.1}-x;n_{1.},p_1,p_2)= \tag{5.5}$$

$$\binom{n_{1.}}{x} p_1^x(1-p_1)^{n_{1.}-x}\binom{n_{1.}}{n_{.1}-x} p_2^{n_{.1}-x} (1-p_2)^{n_{1.}-n_{.1}+x}$$

To examine the nature of the hypothesis being tested, one can reparameterize this into the logistic model. This formulation was first suggested by Cox (1958) and is further treated Cox (1970) and Gart (1971). In the reparameterized model,

$$p_1=\frac{\exp(\beta+\lambda/2)}{1+\exp(\beta+\lambda/2)} \qquad p_2=\frac{\exp(\beta-\lambda/2)}{1+\exp(\beta-\lambda/2)} . \tag{5.6}$$

From this it is easy to see that the odds ratio is

$$t=\frac{p_1(1-p_1)}{p_2(1-p_2)}=\exp(\lambda). \tag{5.7}$$

Thus $\lambda=\ell n(p_1/1-p_1)-\ell n(p_2/1-p_2))$ is the difference in logits and is also known as the log odds ratio. The probabilities p_1 and p_2 are equal when t=1, implying $\lambda=0$. Using this form, the joint probability function of x and $n_{.1}-x$, when $n_{1.}$ pairs have been sampled is

$$P(x,n_{.1}-x;n_{1.},\lambda,\beta)=$$

$$\frac{\binom{n_{1.}}{x}\binom{n_{1.}}{n_{.1}-x}\exp((\lambda/2)(2x-n_{.1})+\beta n_{.1})}{(1+\exp(\beta+\lambda/2))^{n_{1.}}\cdot(1+\exp(\beta-\lambda/2))^{n_{1.}}}. \tag{5.8}$$

The degree of inequality of p_1 and p_2 is expressed in terms of the parameter λ. The parameter β is related to the actual parameter values p_1 and p_2. In (5.8), $n_{.1}$ is the sum of the successes from both populations. The quantity $2x-n_{.1}$ is the difference between the number of successes observed from populations 1 and 2. It can be seen in (5.8) that although the probability function cannot be completely factored, factorization of the numerator (the denominator is not subject to random variation) shows that $2x-n_{.1}$ and $n_{.1}$ are sufficient for λ and β respectively.

It is desired to make inferences on λ, the log odds ratio. The quantity $n_{.1}$ is therefore an ancillary statistic for β. When making inferences about λ, it is proper to consider the conditional

distribution of $2x-n_{.1}$ (or x itself--the distributions are equivalent) given the observed value $n_{.1}$. This conditional distribution is the extended hypergeometric distribution shown (4.7), except that t is now equal to the odds ratio rather than the cross product ratio. The null distribution is again the hypergeometric shown in (4.8), which does not depend on the nuisance parameter β.

To conduct the sequential tests for this case, an observation is chosen at random from each of the two populations at each trial. The sequential test rules are similar to those shown in (3.16), with the modification that $n_{.1}=n/2$ is really the trial number and each trial consists of one observation from each population.

For this case, the sequential test rules are:

$$\begin{aligned} &\text{accept } H_0 \text{ if} && x \le c_L(n_{1.}, n_{.1}) \\ &\text{accept } H_1 \text{ if} && x \ge c_U(n_{1.}, n_{.1}) \end{aligned} \qquad (5.9)$$

and otherwise continue the test and take another sample. These critical values are based on the theory developed in Section 4.3 and are thus found as

$$\begin{aligned} c_L(n_{1.}, n_{.1}) &= \left[g^{-1}(b, n_{1.}, n_{.1}, 2n_{1.}, t_0, t_1) \right] \\ &= \left[(b+F(t_0)-F(t_1))/\ell n(t_1)-\ell n(t_0)) \right] \qquad (5.10) \\ c_U(n_{1.}, n_{.1}) &= \left[g^{-1}(a, n_{1.}, n_{.1}, 2n_{1.}, t_0, t_1) \right] + 1 \\ &= \left[(a+F(t_0)-F(t_1))/(\ell n(t_1)-\ell n(t_0)) \right] + 1 \end{aligned}$$

using the same notation as in (4.17). Here there are only $2n_{1.}+1$ possible values for $n_{.1}$ at each trial $n_{1.}$. That is, $0 \le n_{.1} \le 2n_{1.}$. Tables of the test procedure critical values will therefore be much smaller than those of the cases considered earlier. A numerical example for this case follows.

The hypothesis of equal probability of success for the two populations is specified as

$$H_0: \ t=t_0=1$$
$$\text{versus } H_1: \ t=t_1=5 \qquad (5.11)$$

The desired α and β error probabilities are chosen to be 0.025 and 0.2 respectively. The computer program in the Appendix was used to generate the table of critical values, defining the test rules, shown in Table 5.1. These tests are truncated as in the previous tests presented here. The method of finding the exact properties of this sequential test procedure is given in the next section and the sequential test region shown in Table 5.1 is evaluated there as a numerical example.

In order to conduct such a test, at each trial one selects an item at random from each of the two populations. A score is kept of the cumulative number of successes in both populations. One then compares at each trial the total number of successes in population 1 with the critical value for the corresponding margin totals, using the test rules in (5.9). A numerical example of this procedure follows.

Table 5.1
Critical Values for p_1-p_2 Example

T0= 1.00
T1= 5.00
ALPHA= 0.025
BETA= 0.200

TRIAL NO.	N1. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
	45	46	47	48	49	50									
1	-1.3	-1.3	0,4												
2	-1.3	-1.3	0,4	0,4	1,5										
3	-1.3	-1.3	0,4	0,5	1,5	1,5	2,6								
4	-1.3	-1.3	0,4	0,5	1,5	1,6	2,6	2,6	3,7						
5	-1.3	-1.3	0,4	0,5	1,5	2,6	2,6	2,7	3,7	3,7	4,8				
6	-1.3	-1.3	0,4	0,5	1,5	2,6	2,6	3,7	3,7	3,8	4,8	4,8	5,9		
7	-1.3	-1.3	0,4	0,5	1,5	2,6	2,6	3,7	3,7	4,8	4,8	4,9	5,9	5,9	6,10
8	-1.3	-1.3	0,4	1,5	1,5	2,6	2,6	3,7	3,7	4,8	4,8	5,9	5,9	6,10	6,10
	6,10	7,11													
9	-1.3	-1.3	0,4	1,5	1,5	2,6	2,6	3,7	3,8	4,8	4,9	5,9	5,9	6,10	6,10
	7,11	7,11	7,11	8,12											
10	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	5,9	6,10	6,10	6,11
	7,11	7,11	8,12	8,12	8,12	9,13									
11	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	5,9	6,10	6,10	7,11
	7,11	7,12	8,12	8,12	9,13	9,13	9,13	10,14							
12	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	5,9	6,10	6,10	7,11
	7,11	8,12	8,12	8,13	9,13	9,13	10,14	10,14	10,14	11,15					
13	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	5,9	6,10	6,10	7,11
	7,11	8,12	8,12	9,13	9,13	9,14	10,14	10,14	11,15	11,15	11,15	12,16			
14	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	5,10	6,10	6,11	7,11
	7,12	8,12	8,13	9,13	9,13	10,14	10,14	10,15	11,15	11,15	12,16	12,16	12,16	13,17	
15	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	5,10	6,10	7,11	7,11
	8,12	8,12	9,13	9,13	9,14	10,14	10,14	11,15	11,15	11,16	12,16	12,16	13,17	13,17	13,17
	14,18														
16	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	5,10	6,10	7,11	7,11
	8,12	8,12	9,13	9,13	10,14	10,14	10,15	11,15	11,15	12,16	12,16	12,17	13,17	13,17	14,18
	14,19	14,18	15,19												
17	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	6,10	6,10	7,11	7,11
	8,12	8,12	9,13	9,13	10,14	10,14	11,15	11,15	12,16	12,16	12,16	13,17	13,17	13,18	14,18
	14,18	15,19	15,19	15,19	16,20										
18	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,8	5,9	6,10	6,10	7,11	7,11
	8,12	8,12	9,13	9,13	10,14	10,14	11,15	11,15	12,16	12,16	13,17	13,17	13,17	14,18	14,18
	14,19	15,19	15,19	16,20	16,20	16,20	17,21								
19	-1.3	-1.3	0,4	1,5	1,5	2,6	2,7	3,7	4,8	4,9	5,9	6,10	6,10	7,11	7,11
	8,12	8,13	9,13	9,14	10,14	10,15	11,15	11,16	12,16	12,16	13,17	13,17	14,18	14,18	14,19
	15,19	15,19	15,20	16,20	16,20	17,21	17,21	17,21	18,22						
20	-1.3	-1.3	0,4	1,5	1,5	2,6	3,7	3,7	4,8	4,9	5,9	6,10	6,10	7,11	7,11
	8,12	8,13	9,13	10,14	10,14	11,15	11,15	12,16	12,16	12,17	13,17	13,17	14,18	14,18	15,19
	15,19	15,20	16,20	16,20	17,21	17,21	17,21	18,22	18,22	18,22	19,23				
21	-1.3	-1.3	0,4	1,5	1,5	2,6	3,7	3,7	4,8	4,9	5,9	6,10	6,10	7,11	7,12
	8,12	8,13	9,13	10,14	10,14	11,15	11,15	12,16	12,16	13,17	13,17	13,18	14,18	14,19	15,19
	15,19	15,20	16,20	16,21	17,21	17,21	18,22	18,22	18,22	19,23	19,23	19,23	20,24		
22	-1.3	-1.3	0,4	1,5	1,5	2,6	3,7	3,7	4,8	4,9	5,9	6,10	6,10	7,11	7,12
	8,12	9,13	9,13	10,14	10,14	11,15	11,15	12,16	12,16	13,17	13,17	14,18	14,18	15,19	15,19
	15,20	16,20	16,20	17,21	17,21	17,22	18,22	18,22	19,23	19,23	19,23	20,24	20,24	20,24	21,25
23	-1.3	-1.3	0,4	1,5	1,5	2,6	3,7	3,7	4,8	4,9	5,9	6,10	6,10	7,11	7,12
	8,12	9,13	9,13	10,14	10,14	11,15	11,15	12,16	12,16	13,17	13,17	14,18	14,18	15,19	15,19
	16,20	16,20	16,21	17,21	17,21	18,22	18,22	18,23	19,23	19,23	20,24	20,24	20,24	21,25	21,25
	21,25	22,26													
24	-1.3	-1.3	0,4	1,5	1,5	2,6	3,7	3,7	4,8	4,9	5,9	6,10	6,10	7,11	7,12
	8,12	9,13	9,13	10,14	10,14	11,15	11,16	12,16	12,17	13,17	13,18	14,18	14,19	15,19	15,19
	16,20	16,20	17,21	17,21	17,22	18,22	18,22	19,23	19,23	19,24	20,24	20,24	21,25	21,25	21,25
	22,26	22,26	22,26	23,27											
25	0,1	1,2	1,2	2,3	3,4	3,4	4,5	5,6	5,6	6,7	7,7	7,8	8,9	8,9	9,10
	10,11	10,11	11,12	11,12	12,13	12,13	13,14	14,15	14,15	15,16	15,16	16,17	16,17	17,18	17,18
	17,18	18,19	18,19	19,20	19,20	20,21	20,21	20,21	21,22	21,22	22,23	22,23	22,23	23,24	23,24
	23,24	24,25	24,25	24,25	25,26	25,26									

Table 5.2 contains a typical sequential sample which might be obtained using the above test procedure. Here a 1 represents a success and a 0 represents a failure.

Table 5.2
Typical Sequential Sample

TRIAL	POPULATION 1	POPULATION 2	$n_{.1}$	x
1	1	1	2	1
2	1	0	3	2
3	0	1	4	2
4	0	0	4	2
5	1	0	5	3
6	0	1	6	3
7	1	1	8	4
8	0	1	9	4

The results of this sample at trial 8 (i.e., after 8 pairs have been observed) are summarized in Figure 5.3.

4	4	8
5	3	8
9	7	16

Figure 5.3 Summary of Data From Sample Sequential Test

From examination of the critical values in Table 5.1, it is seen that x=4 is a lower critical value when $n_{.1}=9$; therefore, the test is terminated there and a decision is made in favor of H_0.

The three decision test procedure to test the hypotheses

$$\begin{aligned} H_0&: \ t=t_1<t_0 \\ \text{versus } H_1&: \ t=t_0 \\ \text{versus } H_2&: \ t=t_1>t_0 \end{aligned} \tag{5.12}$$

is similar to that given in Section 3.4.

Again, two SPRTs are used simultaneously; therefore two sets of tables like those in Table 5.1 are computed. The rules for carrying out the sequential test are:

$$\begin{aligned} &\text{accept } H_1 \text{ if} && x\le c_L(n_{1.},n_{.1}) \\ &&\text{and} \quad & x\le d_L(n_{1.},n_{.1}) \\ &\text{accept } H_0 \text{ if} && x\ge c_U(n_{1.},n_{.1}) \\ &&\text{and} \quad & x\le d_L(n_{1.},n_{.1}) \\ &\text{accept } H_2 \text{ if} && x\ge c_U(n_{1.},n_{.1}) \\ &&\text{and} \quad & x\ge d_U(n_{1.},n_{.1}) \end{aligned} \tag{5.13}$$

where $c_L(\cdot)$ and $c_U(\cdot)$ are the lower and upper limits for SPRT1 (i.e., for the first pair of hypotheses in (5.12)) and $d_L(\cdot)$ and $d_U(\cdot)$ are the upper and lower limits for SPRT2 (i.e., for the second pair of hypotheses in (5.12)). These limits are found in a manner analogous to that in Section 3.4, using (5.10).

As a numerical example, consider testing the hypotheses

$$\begin{aligned} H_1&: \ t=t_1=0.2 \\ \text{versus } H_0&: \ t=t_0=1.0 \\ \text{versus } H_2&: \ t=t_2=5.0 \end{aligned} \tag{5.14}$$

The desired error probabilities are chosen to be $\alpha_1=\alpha_2=0.025$ and $\beta_1=\beta_2=0.2$. It is again necessary to generate two sets of critical values, one each for testing between H_1 and H_0 and between H_0 and H_2. The first set of critical values is shown in Table 5.3; the second set is the same as was used in the previous two decision numerical example and is shown in Table 5.1. The test region is again truncated as before. The procedure for carrying out such a test is as explained above, using the rules in (5.13).

5.3 EVALUATION OF THE SEQUENTIAL TEST REGIONS

This section describes the method used to find the exact test properties of the sequential test regions developed in the last section. The direct method of sequential analysis is used in a manner similar to that of Section 3.5. Because the underlying probability model and the test procedure are different, there are some changes. These are outlined below.

At each trial, one observation is taken from each population on the right-hand margin. Let x and $n_{.1}-x$ denote the number of successes observed in populations 1 and 2 respectively at trial $n_{1.}$. From each point $(x,n_{.1}-x,n_{1.})$ in the sample space at each trial $n_{1.}$, there are four possible outcomes at trial $n_{1.}+1$. They are $(x+1,n_{.1}-x+1,n_{1.}+1)$, $(x+1,n_{.1}-x,n_{1.}+1)$, $(x,n_{.1}-x.n_{1.}+1)$, and $(x,n_{.1}-x+1,n_{1.}+1)$. The probabilities of each of these occurrences are shown in Figure 5.4.

Table 5.3
Critical Values for the
Three Decision Test Example

		N= 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
T0=	0.20	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T1=	1.00	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
ALPHA=	0.200	45	46	47	48	49	50									
BETA=	0.025															

TRIAL No.															
1	-3, 1	-2, 2	-2, 2												
2	-3, 1	-2, 2	-2, 2	-1, 3	-1, 3										
3	-3, 1	-2, 2	-2, 2	-2, 3	-1, 3	0, 4	0, 4								
4	-3, 1	-2, 2	-2, 2	-2, 3	-1, 3	±1, 4	0, 4	1, 5	1, 5						
5	-3, 1	-2, 2	-2, 2	-2, 3	-1, 3	±1, 3	0, 4	0, 5	1, 5	2, 6	2, 6				
6	-3, 1	-2, 2	-2, 2	-2, 3	-1, 3	±1, 3	0, 4	0, 4	1, 5	1, 6	2, 6	3, 7	3, 7		
7	-3, 1	-2, 2	-2, 2	-2, 3	-1, 3	±1, 3	0, 4	0, 4	1, 5	1, 5	2, 6	2, 7	3, 7	4, 8	4, 8
8	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	0, 4	0, 4	1, 5	1, 5	2, 6	2, 6	3, 7	3, 7	4, 8
	5, 9	5, 9													
9	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	0, 4	0, 4	0, 5	1, 5	1, 6	2, 6	3, 7	3, 7	4, 8
	4, 8	5, 9	6,10	6,10											
10	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	2, 6	2, 6	3, 7	3, 8
	4, 8	5, 9	5, 9	6,10	7,11	7,11									
11	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	-1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	2, 6	2, 6	3, 7	3, 7
	4, 8	4, 9	5, 9	6,10	6,10	7,11	8,12	8,12							
12	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	-1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	2, 6	2, 6	3, 7	3, 7
	4, 8	4, 8	5, 9	5,10	6,10	7,11	7,11	8,12	9,13	9,13					
13	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	2, 6	2, 6	3, 7	3, 7
	4, 8	4, 8	5, 9	5, 9	6,10	6,11	7,11	8,12	8,12	9,13	10,14	10,14			
14	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	1, 6	2, 6	2, 7	3, 7
	3, 8	4, 8	4, 9	5, 9	6,10	6,10	7,11	7,12	8,12	9,13	9,13	10,14	11,15	11,15	
15	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	1, 6	2, 6	2, 6	3, 7
	3, 7	4, 8	4, 8	5, 9	5,10	6,10	7,11	7,11	8,12	8,13	9,13	10,14	10,14	11,15	12,16
	12,16														
16	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	-1, 3	±1, 4	0, 4	0, 4	1, 5	1, 5	1, 6	2, 6	2, 6	3, 7
	3, 7	4, 8	4, 8	5, 9	5, 9	6,10	6,11	7,11	8,12	8,12	9,13	9,14	10,14	11,15	11,15
	12,16	13,17	13,17												
17	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	-1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	1, 5	2, 6	2, 6	3, 7
	3, 7	4, 8	4, 8	5, 9	5, 9	6,10	6,10	7,11	7,11	8,12	9,13	9,13	10,14	10,15	11,15
	12,16	12,16	13,17	14,18	14,18										
18	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 4	0, 4	0, 4	1, 5	1, 5	1, 5	2, 6	2, 6	3, 7
	3, 7	4, 8	4, 8	5, 9	5, 9	6,10	6,11	7,11	7,11	8,12	8,12	9,13	10,14	10,14	11,15
	11,16	12,16	13,17	13,17	14,18	15,19	15,19								
19	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	-1, 3	-1, 4	0, 4	0, 4	0, 5	1, 5	1, 5	2, 6	2, 6	3, 7
	3, 7	3, 8	4, 8	4, 9	5, 9	5,10	6,10	6,11	7,11	8,12	8,12	9,13	9,13	10,14	10,15
	11,15	12,16	12,17	13,17	14,18	14,18	15,19	16,20	16,20						
20	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	-1, 3	-1, 4	0, 4	0, 4	0, 5	1, 5	1, 5	2, 6	2, 6	3, 7
	3, 7	3, 8	4, 8	4, 8	5, 9	5, 9	6,10	6,10	7,11	7,12	8,12	9,13	9,13	10,14	10,14
	11,15	11,16	12,16	13,17	13,18	14,18	15,19	15,19	16,20	17,21	17,21				
21	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 3	0, 4	0, 4	0, 5	1, 5	1, 5	2, 6	2, 6	2, 7
	3, 7	3, 8	4, 8	4, 8	5, 9	5, 9	6,10	6,10	7,11	7,11	8,12	8,13	9,13	9,14	10,14
	11,15	11,15	12,16	12,17	13,17	14,18	14,18	15,19	16,20	16,20	17,21	18,22	18,22		
22	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 3	0, 4	0, 4	0, 5	1, 5	1, 5	2, 6	2, 6	2, 7
	3, 7	3, 7	4, 8	4, 8	5, 9	5, 9	6,10	6,10	7,11	7,11	8,12	8,12	9,13	9,13	10,14
	10,15	11,15	12,16	12,16	13,17	13,18	14,18	15,19	15,19	16,20	17,21	17,21	18,22	19,23	19,23
23	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	-1, 3	-1, 3	0, 4	0, 4	0, 5	1, 5	1, 5	2, 6	2, 6	2, 7
	3, 7	3, 7	4, 8	4, 8	5, 9	5, 9	6,10	6,10	7,11	7,11	8,12	8,12	9,13	9,13	10,14
	10,14	11,15	11,16	12,16	13,17	13,17	14,18	14,19	15,19	16,20	16,20	17,21	18,22	18,22	19,23
	20,24	20,24													
24	-3, 1	-2, 2	-2, 2	-2, 2	-1, 3	±1, 3	-1, 3	0, 4	0, 4	0, 5	1, 5	1, 5	2, 6	2, 6	2, 7
	3, 7	3, 7	4, 8	4, 8	5, 9	5, 9	5,10	6,10	6,11	7,11	7,12	8,12	8,13	9,13	10,14
	10,14	11,15	11,15	12,16	12,17	13,17	14,18	14,18	15,19	15,20	16,20	17,21	17,21	18,22	19,23
	19,23	20,24	21,25	21,25											
25	-1, 0	-1, 0	0, 1	0, 1	0, 1	1, 2	1, 2	1, 2	1, 2	2, 3	2, 3	2, 3	3, 4	3, 4	4, 5	4, 5
	4, 5	5, 6	5, 6	6, 7	6, 7	7, 8	7, 8	7, 8	8, 9	8, 9	9,10	9,10	10,11	10,11	11,12
	12,13	12,13	13,14	13,14	14,15	14,15	15,16	16,17	16,17	17,18	17,18	18,19	19,20	19,20	20,21
	21,22	21,22	22,23	23,24	23,24	24,25									

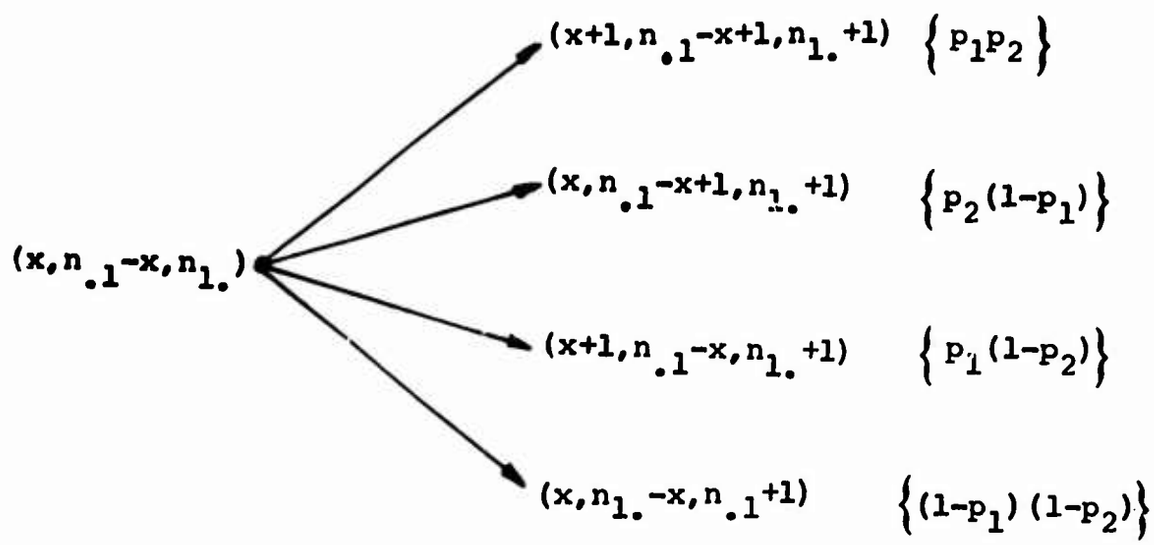

Figure 5.4 Possible Outcomes at Each Trial

The sequential test begins at trial 0 where the only possibility is $(x=0, n_{.1}-x=0, n_{1.}=0)$, which therefore has a probability of 1. The probabilities of reaching each point $(x, n_{.1}-x, n_{1.})$ at trial $n_{1.}$ is then computed recursively for $n_{1.}=1,2,\ldots,n_0$ starting with the point at the origin. The probability of reaching each point inside or on the boundary of the sequential test region is a function of the true state of nature, which is completely specified by p_1 and p_2. The OC and ASN functions and the distribution of the DSN will therefore be functions of these two parameters.

Starting at trial 0, the probabilities of reaching each point $(x, n_{.1}-x, n_{1.})$ for a specified state of nature are computed using the multiplication rule and summing the probabilities of all the different ways one might reach a point in the space. For example, the point $(x, n_{.1}-x, n_{1.})$ at trial $n_{1.}$ could have been reached from one of four points at trial $n_{1.}-1$, that is, from $(x-1, n_{.1}-x-1, n_{1.}-1)$, $(x, n_{.1}-x-1, n_{1.}-1)$, $(x-1, n_{.1}-x, n_{1.}-1)$ or from $(x, n_{1.}-x, n_{1.}-1)$. Let Ai_n denote the event of

of accepting hypothesis H_i=0,1 and C_n the event of being in the continuation region at trial n, that is,

$$\begin{aligned} A0_n &= \{(x,n_{.1}-x,n_{1.}) \mid x \geq c_L(n_{1.},n_{.1})\} \\ A1_n &= \{(x,n_{.1}-x,n_{1.}) \mid x \leq c_U(n_{1.},n_{.1})\} \\ C_n &= \{(x,n_{.1}-x,n_{1.}) \mid c_L(n_{1.},n_{.1}) < x < c_U(n_{1.},n_{.1})\} \end{aligned} \tag{5.15}$$

The recursive formula used to find the probabilities for each point in the $(x,n_{.1}-x,n_{1.})$ space is

$$\begin{aligned} &P_S(x,n_{.1}-x,n_{1.};p_1,p_2) = \\ &I(x-1,n_{.1}-x-1,n_{1.}-1)P_S(x-1,n_{.1}-x-1;n_{1.}-1,p_1,p_2)(p_1)(p_2) \\ &+I(x,n_{.1}-x-1,n_{1.}-1)P_S(x,n_{.1}-x-1;n_{1.}-1,p_1,p_2)(1-p_1)p_2 \\ &+I(x-1,n_{.1}-x.n_{1.}-1)P_S(x-1,n_{.1}-x;n_{1.}-1,p_1,p_2)p_1(1-p_2) \\ &+I(x,n_{.1}-x,n_{1.}-1)P_S(x,n_{.1}-x;n_{1.}-1,p_1,p_2)(1-p_1)(1-p_2) \end{aligned} \tag{5.16}$$

where $$P_S(x,n_{1.}-x;0,p_1,p_2) = \begin{cases} 1 & \text{if } x=n_{1.}=n_{.1}=0 \\ 0 & \text{otherwise} \end{cases}$$

and $$I(x,n_{1.}-x,n_{1.}) = \begin{cases} 1 & \text{if } (x,n_{.1}-x,n_{1.}) \in C_n \\ 0 & \text{otherwise} \end{cases}$$

The indicator function I accounts for the termination of the test when one of the critical values has been reached. Once again, one need only compute the probabilities for those points inside or on the boundary of the sequential test region; all other points have probability zero. The probabilities of each of the

events $Ai_{n_{1.}}$, i=0,1 and $n_{1.}=1,2,\ldots n_0$ must be computed for each desired state of nature (p_1,p_2). (The computational simplification given in Section 3.5 can again be used here, with some small modification.) The probabilities are computed as follows

$$P(Ai_{n_{1.}},p_1,p_2)= \tag{5.17}$$

$$\sum_{n_{.1}=0}^{2n_{1.}} \sum_{x=IL}^{IU} J_i(x,n_{.1}-x,n_{1.})P_S(x,n_{.1}-x;n_{.1},p_1,p_2)$$

where $IL=MAX(0,n_{.1}-n_{1.})$

$IU=MIN(n_{.1},n_{1.})$

$$J_i(x,n_{.1}-x,n_{1.})= \begin{cases} 1 & \text{if } (x,n_{.1}-x,n_{1.})\varepsilon Ai_{n_{1.}} \\ 0 & \text{otherwise} \end{cases}$$

The indicator function J_i, i=0,1 is used to accumulate all of the probability of accepting H_i. Once these probabilities have been computed, one can find the exact test properties by using the same procedure given in Section 3.5.

The procedure for finding the properties of a three decision test region is analogous to the development in Section 3.6, constructing two SPRTs (i.e., sets of critical values for the test statistic) to be run simultaneously. The sequential test rules are the same as those in (3.25).

As a numerical example, the sequential test regions found in the last section and evaluated here. The hypothesis being tested for the two decision example is

$$H_0: \quad t=t_0=1$$
$$\text{versus } H_1: \quad t=t_1=5 \tag{5.18}$$

with $\alpha=0.025$ and $\beta=0.2$. The sequential test region is shown in Table 5.1; the exact test properties for this truncated test region are shown in Table 5.4.

For the three decision test example, the hypotheses being tested are

$$H_0: \quad t=t_1=0.2$$
$$\text{versus } H_1: \quad t=t_0=1 \tag{5.19}$$
$$\text{versus } H_2: \quad t=t_2=5,$$

with $\alpha_1=\alpha_2=0.025$ and $\beta_1=\beta_2=0.2$. The sequential test regions for this example are shown in Tables 5.1 and 5.3. The exact test properties are shown in Table 5.5.

There are several things which should be noted about the properties of these tests. First, the α error probabilities (i.e., $P(H_1)$ and $P(H_2)$ when t=1, give α'_1 and α'_2 respectively for the three decision test procedure) are somewhat higher than what was specified as the desired probability. Also, the sizes of the error probabilities vary with values of $p_1=p_2$. For example, when $p_1=p_2=0.4$ in Table 5.5, $\alpha'_1=\alpha'_2=0.0459$ and when $p_1=p_2=0.5$, $\alpha'_1=\alpha'_2=0.0396$. The power of the test (i.e., $P(H_0)$ when t=5) also varies over the equal values of t. The table shows clearly that more power can be expected if one of the probabilities is close to 0.5 (again considering equal values of t over the (p_1,p_2) space).

Table 5.4
Two Decision $p_1=p_2$
Example Test Properties

p_1	p_2	t	$P(H_0)$	$P(H_1)$	ASN	$P(C_{n_0-1})$
0.10000	0.10000	1,0	0.89162	0.10838	18.42	0.45817
0.10000	0.05263	2,0	0.71863	0.28137	22.40	0.76284
0.10000	0.03571	3,0	0.61290	0.38710	23.63	0.86948
0.10000	0.02703	4,0	0.54612	0.45388	24.15	0.91677
0.10000	0.02174	5,0	0.50076	0.49924	24.42	0.94153
0.10000	0.01818	6,0	0.46809	0.53191	24.58	0.95602
0.10000	0.01562	7,0	0.44349	0.55651	24.67	0.96519
0.20000	0.20000	1,0	0.92924	0.07076	14.48	0.25443
0.20000	0.11111	2,0	0.70115	0.29885	20.01	0.59500
0.20000	0.07692	3.0	0.52774	0.47226	22.03	0.73372
0.20000	0.05882	4,0	0.41481	0.58519	22.92	0.79312
0.20000	0.04762	5,0	0.33933	0.66067	23.38	0.82104
0.20000	0.04000	6,0	0.28652	0.71348	23.64	0.83494
0.20000	0.03448	7,0	0.24799	0.75201	23.80	0.84199
0.30000	0.30000	1.0	0.94623	0.05377	12.30	0.16144
0.30000	0.17647	2,0	0.70231	0.29769	18.33	0.48263
0.30000	0.12500	3,0	0.49521	0.50479	20.58	0.60788
0.30000	0.09677	4,0	0.36012	0.63988	21.47	0.64254
0.30000	0.07895	5,0	0.27242	0.72758	21.83	0.64484
0.30000	0.06667	6,0	0.21342	0.78658	21.98	0.63572
0.30000	0.05769	7,0	0.17214	0.82786	22.04	0.62291
0.40000	0.40000	1,0	0.95642	0.04358	11.05	0.11744
0.40000	0.25000	2,0	0.70523	0.29477	16.93	0.39305
0.40000	0.18182	3,0	0.47149	0.52851	19.16	0.49473
0.40000	0.14286	4,0	0.32054	0.67946	19.86	0.50134
0.40000	0.11765	5,0	0.22686	0.77314	19.99	0.47714
0.40000	0.10000	6,0	0.16710	0.83290	19.92	0.44595
0.40000	0.08696	7,0	0.12741	0.87259	19.77	0.41557
0.50000	0.50000	1,0	0.96234	0.03766	10.65	0.10643
0.50000	0.33333	2,0	0.71993	0.28007	15.99	0.34105
0.50000	0.25000	3,0	0.46620	0.53380	18.01	0.41490
0.50000	0.20000	4,0	0.30129	0.69871	18.47	0.39892
0.50000	0.16667	5,0	0.20185	0.79815	18.35	0.35727
0.50000	0.14286	6,0	0.14085	0.85915	18.04	0.31357
0.50000	0.12500	7,0	0.10204	0.89796	17.69	0.27468
0.60000	0.60000	1,0	0.95642	0.04358	11.05	0.11744
0.60000	0.42857	2,0	0.73222	0.26778	15.71	0.32836
0.60000	0.33333	3,0	0.48188	0.51812	17.38	0.37880
0.60000	0.27273	4,0	0.30535	0.69465	17.55	0.34308
0.60000	0.23077	5.0	0.19715	0.80285	17.18	0.28751
0.60000	0.20000	6,0	0.13211	0.86789	16.66	0.23569
0.60000	0.17647	7,0	0.09211	0.90789	16.14	0.19307
0.70000	0.70000	1,0	0.94623	0.05377	12.30	0.16144
0.70000	0.53846	2,0	0.71230	0.28770	16.28	0.35607
0.70000	0.43750	3,0	0.47531	0.52469	17.54	0.38707
0.70000	0.36842	4,0	0.30923	0.69077	17.40	0.33577
0.70000	0.31818	5,0	0.20265	0.79735	16.76	0.26854
0.70000	0.28000	6,0	0.13580	0.86420	16.02	0.20881
0.70000	0.25000	7,0	0.09372	0.90628	15.32	0.16144
0.80000	0.80000	1,0	0.92924	0.07076	14.48	0.25443
0.80000	0.66667	2,0	0.70235	0.29765	17.82	0.44940
0.80000	0.57143	3,0	0.46793	0.53207	18.79	0.46799
0.80000	0.50000	4,0	0.30129	0.69871	18.47	0.39892
0.80000	0.44444	5,0	0.19647	0.80353	17.63	0.31202
0.80000	0.40000	6,0	0.13211	0.86789	16.66	0.23569
0.80000	0.36364	7,0	0.09192	0.90808	15.74	0.17622
0.90000	0.90000	1,0	0.89162	0.10838	18.42	0.45817
0.90000	0.81818	2,0	0.70037	0.29963	20.37	0.61855
0.90000	0.75000	3,0	0.51075	0.48925	21.31	0.67053
0.90000	0.69231	4,0	0.35654	0.64346	21.34	0.63078
0.90000	0.64286	5,0	0.24415	0.75585	20.78	0.54425
0.90000	0.60000	6.0	0.16710	0.83290	19.92	0.44595
0.90000	0.56250	7,0	0.11569	0.88431	18.94	0.35451

Table 5.5
Three Decision $p_1=p_2$
Example Test Properties

p_1	p_2	t	$P(H_1)$	$P(H_0)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.10000	0.10000	1,0	0.10893	0.78214	0.10893	24.63	0.89490
0.10000	0.05263	2,0	0.03982	0.68234	0.28184	24.88	0.96410
0.10000	0.03571	3,0	0.01840	0.59418	0.38743	24.94	0.98260
0.10000	0.02703	4,0	0.01133	0.53456	0.45410	24.96	0.98879
0.10000	0.02174	5,0	0.00772	0.49287	0.49941	24.97	0.99128
0.10000	0.01818	6,0	0.00561	0.46236	0.53204	24.97	0.99241
0.10000	0.01562	7,0	0.00426	0.43913	0.55661	24.97	0.99296
0.20000	0.20000	1,0	0.07373	0.85254	0.07373	22.52	0.59699
0.20000	0.11111	2,0	0.01452	0.67997	0.30551	23.94	0.78741
0.20000	0.07692	3,0	0.00587	0.51533	0.47881	24.32	0.85868
0.20000	0.05882	4,0	0.00319	0.40606	0.59075	24.42	0.88042
0.20000	0.04762	5,0	0.00203	0.33273	0.66525	24.43	0.88525
0.20000	0.04000	6,0	0.00142	0.28134	0.71725	24.42	0.88406
0.20000	0.03448	7,0	0.00105	0.24380	0.75515	24.40	0.88074
0.30000	0.30000	1,0	0.05686	0.88629	0.05686	19.53	0.36366
0.30000	0.17647	2,0	0.00616	0.68306	0.31078	22.04	0.59731
0.30000	0.12500	3,0	0.00165	0.47715	0.52120	22.91	0.69534
0.30000	0.09677	4,0	0.00069	0.34365	0.65566	23.05	0.71037
0.30000	0.07895	5,0	0.00037	0.25804	0.74159	22.98	0.69806
0.30000	0.06667	6,0	0.00023	0.20106	0.79871	22.85	0.67817
0.30000	0.05769	7,0	0.00015	0.16154	0.83830	22.71	0.65737
0.40000	0.40000	1,0	0.04592	0.90817	0.04592	17.65	0.25677
0.40000	0.25000	2,0	0.00336	0.68783	0.30882	20.05	0.45427
0.40000	0.18182	3,0	0.00060	0.45009	0.54931	21.14	0.54776
0.40000	0.14286	4,0	0.00018	0.29854	0.70129	21.25	0.54709
0.40000	0.11765	5,0	0.00007	0.20643	0.79350	21.02	0.51551
0.40000	0.10000	6,0	0.00004	0.14891	0.85106	20.71	0.47786
0.40000	0.08696	7,0	0.00002	0.11147	0.88851	20.40	0.44217
0.50000	0.50000	1,0	0.03958	0.92085	0.03958	17.07	0.23180
0.50000	0.33333	2,0	0.00219	0.70465	0.29316	18.83	0.38289
0.50000	0.25000	3,0	0.00030	0.44469	0.55501	19.76	0.45092
0.50000	0.20000	4,0	0.00007	0.27806	0.72187	19.69	0.43042
0.50000	0.16667	5,0	0.00002	0.17978	0.82020	19.25	0.38388
0.50000	0.14286	6,0	0.00001	0.12096	0.87903	18.73	0.33575
0.50000	0.12500	7,0	0.00000	0.08450	0.91550	18.24	0.29316
0.60000	0.60000	1,0	0.04592	0.90817	0.04592	17.65	0.25677
0.60000	0.42857	2,0	0.00182	0.71801	0.28017	18.50	0.36713
0.60000	0.33333	3,0	0.00021	0.46102	0.53877	19.07	0.41043
0.60000	0.27273	4,0	0.00004	0.28189	0.71807	18.71	0.36944
0.60000	0.23077	5,0	0.00001	0.17443	0.82556	18.03	0.30861
0.60000	0.20000	6,0	0.00000	0.11153	0.88846	17.31	0.25239
0.60000	0.17647	7,0	0.00000	0.07399	0.92601	16.65	0.20634
0.70000	0.70000	1,0	0.05686	0.88629	0.05686	19.53	0.36366
0.70000	0.53846	2,0	0.00254	0.69621	0.30125	19.19	0.40247
0.70000	0.43750	3,0	0.00023	0.45418	0.54559	19.24	0.41947
0.70000	0.36842	4,0	0.00004	0.28582	0.71414	18.56	0.36170
0.70000	0.31818	5,0	0.00001	0.17975	0.82024	17.62	0.28864
0.70000	0.28000	6,0	0.00000	0.11478	0.88521	16.67	0.22411
0.70000	0.25000	7,0	0.00000	0.07501	0.92498	15.82	0.17305
0.80000	0.80000	1,0	0.07373	0.85254	0.07373	22.52	0.59699
0.80000	0.66667	2,0	0.00492	0.68353	0.31155	21.32	0.54182
0.80000	0.57143	3,0	0.00047	0.44637	0.55315	20.69	0.51438
0.80000	0.50000	4,0	0.00007	0.27806	0.72187	19.69	0.43042
0.80000	0.44444	5,0	0.00002	0.17400	0.82598	18.49	0.33481
0.80000	0.40000	6,0	0.00000	0.11153	0.88846	17.31	0.25239
0.80000	0.36364	7,0	0.00000	0.07355	0.92645	16.24	0.18854
0.90000	0.90000	1,0	0.10893	0.78214	0.10893	24.63	0.89490
0.90000	0.81818	2,0	0.01729	0.67792	0.30479	24.19	0.82273
0.90000	0.75000	3,0	0.00308	0.49590	0.50102	23.71	0.77808
0.90000	0.69231	4,0	0.00062	0.33945	0.65993	22.92	0.69684
0.90000	0.64286	5,0	0.00014	0.22567	0.77419	21.88	0.58900
0.90000	0.60000	6,0	0.00004	0.14891	0.85106	20.71	0.47786
0.90000	0.56250	7,0	0.00001	0.09880	0.90119	19.54	0.37788

It is interesting to note that the deviations of actual test properties from the desired test properties over the range of p_1 and p_2 are remarkably small (when compared with other truncated sequential and fixed size tests of such hypotheses). This is especially true when the probability of continuation to trial n_0 (i.e., $P(C_{n_0-1})$) is not too large.

If the actual error probabilities are not satisfactory, there are two possible solutions to the problem. First, modification of the truncation rules at trial n_0 can be used to adjust the error probabilities, as discussed in Section 3.5. Also, one might try using different values for a and b, which are used in (4.15) to develop the sequential test regions. The exact test properties are easy enough to compute (especially in this special case), so that one can use a trial and error method to obtain the desired results. An example of this procedure is given in the next section, along with a comparison of the tests given here with other tests which have been proposed for testing the same hypotheses. Further discussion of this topic is contained in Chapter 7.

5.4 FURTHER NUMERICAL EXAMPLES AND COMPARISON WITH OTHER SIMILAR TESTS

This section presents two further numerical examples. The first example is compared with a similar fixed size test; the second is compared with a similar sequential test.

Table 5.6 gives the test properties of a sequential test for the equality of two unknown binomial proportions, testing the hypotheses

$$H_1: \ t=t_1=1/9.3333$$
$$\text{versus } H_0: \ t=t_0=1 \qquad (5.20)$$
$$\text{versus } H_2: \ t=t_2=9.3333$$

with specified desired error probabilities $\alpha_1=\alpha_2=0.023$ and $\beta_1=\beta_2=0.35$. The test is truncated at trial 25. The sequential test region for this test can be found, as explained above, by using the computer program given in the Appendix. Table 5.7 compares the power and ASN functions of this test with the power function of the UMPUT fixed size test with sample size $n^*=15$ (i.e., 15 pairs are sampled). The power of the latter is given by Harkness (1959).

The α error probability for the UMPUT test is 0.05 for all values of $p_1=p_2$. For the sequential test, the α error probabilities vary (for the combinations of p_1 and p_2 shown in the table) from 0.079 (when $p_1=p_2=0.1$) to 0.045 (when $p_1=p_2=0.5$). These probabilities are close enough to 0.05 to facilitate comparisons. The power of the sequential test (again for the points shown in the table) is seen to be uniformly higher than that of the fixed size sample test. Also, for most of these points, the ASN function is less than 15, the sample size of the fixed size test. For some combinations of the values of p_1 and p_2, the ASN exceeds 15. This occurs when p_1 and/or p_2 approach the extreme values of 0 and 1.

Table 5.6
Three Decision $p_1=p_2$
Example Test Properties

P_1	P_2	t	$P(H_1)$	$P(H_0)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.10000	0.10000	1,0	0.07391	0.85218	0.07391	19,96	0.42634
0.10000	0.01000	9,3	0.00249	0.43385	0.56366	24,11	0.87071
0.10000	0.01000	20,0	0.00060	0.35319	0.64621	24,51	0.92368
0.20000	0.20000	1,0	0.03957	0.92089	0.03953	14,44	0.15615
0.20000	0.10000	2,2	0.00585	0.75036	0.24379	18,42	0.37453
0.20000	0.03000	9.3	0.00053	0.27803	0.72144	22,41	0.65125
0.20000	0.01000	20,0	0.00014	0.14939	0.85047	22,77	0.66791
0.30000	0.30000	1.0	0.02776	0.94477	0.02747	11,03	0.05669
0.30000	0.20000	1.7	0.00537	0.86528	0.12935	13,26	0.13772
0.30000	0.10000	3,9	0.00039	0.52511	0.47451	17,46	0.33221
0.30000	0.04000	9,3	0.00005	0.20117	0.79878	19,21	0.37420
0.30000	0.02000	20.0	0.00001	0.07798	0.92201	19,17	0.32638
0.40000	0.40000	1,0	0.02414	0.95244	0.02342	9,44	0.02744
0.40000	0.30000	1,6	0.00573	0.90627	0.08800	10,64	0.06025
0.40000	0.20000	2,7	0.00083	0.71763	0.28154	13,17	0.15005
0.40000	0.10000	6.0	0.00003	0.32108	0.67889	15,70	0.21608
0.40000	0.07000	9,3	0.00001	0.17220	0.82780	15,91	0.19147
0.40000	0.03000	20,0	0.00000	0.05051	0.94949	15,39	0.12744
0.50000	0.50000	1.0	0.02345	0.95406	0.02250	8,96	0.02084
0.50000	0.40000	1.5	0.00603	0.91810	0.07587	9,56	0.03807
0.50000	0.30000	2.3	0.00125	0.78661	0.21214	11,15	0.08657
0.50000	0.20000	4,0	0.00016	0.52146	0.47837	12,95	0.13282
0.50000	0.10000	9.0	0.00001	0.17633	0.82367	13,37	0.10083
0.50000	0.10000	9.3	0.00000	0.16630	0.83369	13,34	0.09797
0.50000	0.05000	20.0	0.00000	0.04388	0.95612	12,44	0.04612
0.60000	0.60000	1.0	0.02414	0.95244	0.02342	9,44	0.02744
0.60000	0.50000	1.5	0.00603	0.91810	0.07587	9,56	0.03807
0.60000	0.40000	2.2	0.00139	0.80251	0.19611	10,55	0.07084
0.60000	0.30000	3.5	0.00027	0.59350	0.40624	11,73	0.09964
0.60000	0.20000	6,0	0.00003	0.32231	0.67766	12,16	0.08726
0.60000	0.14000	9,3	0.00001	0.16557	0.83442	11,69	0.05500
0.60000	0.07000	20,0	0.00000	0.04303	0.95697	10,44	0.01689
0.60000	0.10000	13.5	0.00000	0.08804	0.91196	11,08	0.03222
0.70000	0.70000	1,0	0.02776	0.94477	0.02747	11,03	0.05669
0.70000	0.60000	1,6	0.00573	0.90627	0.08800	10,64	0.06025
0.70000	0.50000	2,3	0.00125	0.78661	0.21214	11,15	0.08657
0.70000	0.40000	3.5	0.00027	0.59350	0.40624	11,73	0.09964
0.70000	0.30000	5,4	0.00005	0.36616	0.63379	11,74	0.08043
0.70000	0.20000	9,3	0.00001	0.16641	0.83358	10,85	0.03824
0.70000	0.10000	20,0	0.00000	0.04416	0.95584	9,27	0.00711
0.70000	0.10000	21,0	0.00000	0.04038	0.95962	9,18	0.00629
0.80000	0.80000	1.0	0.03957	0.92089	0.03953	14,44	0.15615
0.80000	0.70000	1.7	0.00537	0.86528	0.12935	13,26	0.13772
0.80000	0.60000	2,7	0.00083	0.71763	0.28154	13,17	0.15005
0.80000	0.50000	4,0	0.00016	0.52146	0.47837	12,95	0.13282
0.80000	0.40000	6,0	0.00003	0.32231	0.67766	12,16	0.08726
0.80000	0.30000	9,3	0.00001	0.16641	0.83358	10,85	0.03824
0.80000	0.20000	16,0	0.00000	0.06793	0.93206	9,28	0.00882
0.80000	0.17000	20.0	0.00000	0.04609	0.95391	8,75	0.00432
0.80000	0.10000	36.0	0.00000	0.01592	0.98408	7,72	0.00056
0.90000	0.90000	1,0	0.07391	0.85218	0.07391	19,96	0.42634
0.90000	0.80000	2.2	0.00585	0.75036	0.24379	18,42	0.37453
0.90000	0.70000	3.9	0.00039	0.52511	0.47451	17,46	0.33221
0.90000	0.60000	6,0	0.00003	0.32108	0.67889	15,70	0.21608
0.90000	0.50000	9,0	0.00001	0.17633	0.82367	13,37	0.10083
0.90000	0.49000	9,3	0.00000	0.16616	0.83384	13,15	0.09244
0.90000	0.40000	13,5	0.00000	0.08804	0.91196	11,08	0.03222
0.90000	0.31000	20,0	0.00000	0.04397	0.95603	9,36	0.00767
0.90000	0.30000	21,0	0.00000	0.04038	0.95962	9,18	0.00629
0.90000	0.20000	36,0	0.00000	0.01592	0.98408	7,72	0.00056
0.90000	0.10000	81,0	0.00000	0.00375	0.99625	6,55	0.00001

Table 5.7
Comparison with Fixed Size Test

p_1	p_2	t	$P_{15}(H_a)$	$P_s(H_a)$	ASN
.1	.1	1.00	0.0500	0.1472	19.96
.2	.2	1.00	0.0500	0.0791	14.44
.2	.1	2.25	0.1040	0.2496	18.42
.3	.3	1.00	0.0500	0.0552	11.03
.3	.1	3.85	0.2535	0.4749	17.46
.4	.4	1.00	0.0500	0.0476	9.44
.4	.1	6.00	0.4646	0.6789	15.70
.5	.5	1.00	0.0500	0.0459	8.96
.5	.1	9.00	0.6820	0.8237	13.37
.6	.6	1.00	0.0500	0.0476	9.44
.6	.2	6.00	0.6095	0.6777	12.16
.7	.7	1.00	0.0500	0.0552	11.03
.7	.2	9.33	0.8020	0.8336	10.85
.8	.8	1.00	0.0500	0.0791	14.44
.8	.3	9.33	0.8020	0.8336	10.85
.9	.9	1.00	0.0500	0.1473	19.96
.9	.5	9.00	0.6820	0.8237	13.37

Overall, however, the sequential tests seems superior to the fixed size test.

Table 5.8 gives the test properties of another numerical example for this same problem. The specified desired error probabilities, however, have been changed to $\alpha_1=\alpha_2=0.05$ and $\beta_1=\beta_2=0.2$. The test is still truncated at trial 25. Table 5.9 compares the properties of this test with those of Test Plan #4 from the Ph.D. dissertation of Öksoy (1972). His is also a test for the equality of two unknown binomial proportions but is based on the statistic D=x-y (the difference between the number of successes in the two populations, which is not, in general, a sufficient statistic) and the test is truncated at trial 30.

While uniform superiority cannot be claimed for the new sequential test presented here, it appears that for most points in the (p_1,p_2) parameter space, it will offer considerable advantage. The new test seems to have better properties over a wider range of the values of p_1 and p_2. The test of Oksoy outperforms the new test in two parts of the (p_1,p_2) parameter space. The first is where the differences between p_1 and p_2 are very large (e.g., .5 vs. .9 and .8 vs. .3). The advantage with respect to the power, in this part of the (p_1,p_2) space, however, is not large. The test of Öksoy is also superior with respect to the α error probabilities for values of $p_1=p_2$ which are small. This, however, results in a corresponding loss of power for his test for values of p_1 and p_2 which differ much with respect to the odds ratio, but little with respect to the difference $\Delta=p_1-p_2$ between

Table 5.8
Three Decision $p_1=p_2$
Example Test Properties

P_1	P_2	t	$P(H_1)$	$P(H_0)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.10000	0.10000	1,0	0.10857	0.78285	0.10857	21,72	0.61937
0.10000	0.01000	9.3	0.00253	0.40202	0.59545	24.50	0.91599
0.10000	0.01000	20.0	0.00061	0.33603	0.66336	24,55	0.91927
0.20000	0.20000	1,0	0.06858	0.86284	0.06858	15,78	0.22288
0.20000	0.10000	2,2	0.01145	0.64912	0.33943	19,34	0.44272
0.20000	0.03000	9,3	0.00062	0.19243	0.80696	21,56	0.56303
0.20000	0.01000	20,0	0.00015	0.09952	0.90033	21,42	0.52571
0.30000	0.30000	1,0	0.04932	0.90135	0.04932	12,09	0.08025
0.30000	0.20000	1,7	0.01154	0.79729	0.19117	14,25	0.17222
0.30000	0.10000	3,9	0.00103	0.41986	0.57911	17,33	0.31627
0.30000	0.04000	9,3	0.00009	0.13157	0.86834	17,72	0.27963
0.30000	0.02000	20,0	0.00002	0.04303	0.95695	17,11	0.20810
0.40000	0.40000	1.0	0.04407	0.91186	0.04407	10,39	0.03611
0.40000	0.30000	1,6	0.01275	0.85129	0.13596	11,48	0.07332
0.40000	0.20000	2.7	0.00235	0.62853	0.36912	13,54	0.15923
0.40000	0.10000	6,0	0.00014	0.23957	0.76029	14,82	0.17829
0.40000	0.07000	9.3	0.00003	0.11731	0.88266	14,48	0.13705
0.40000	0.03000	20.0	0.00000	0.02978	0.97022	13,51	0.07306
0.50000	0.50000	1.0	0.04283	0.91435	0.04283	9,93	0.02671
0.50000	0.40000	1.5	0.01385	0.86801	0.11814	10,42	0.04337
0.50000	0.30000	2.3	0.00369	0.71030	0.28601	11,61	0.08925
0.50000	0.20000	4.0	0.00065	0.42926	0.57009	12,64	0.12312
0.50000	0.10000	9.0	0.00004	0.12490	0.87507	12,11	0.07488
0.50000	0.10000	9,3	0.00003	0.11709	0.88287	12,05	0.07189
0.50000	0.05000	20.0	0.00000	0.02790	0.97210	10,88	0.02606
0.60000	0.60000	1.0	0.04407	0.91186	0.04407	10,39	0.03611
0.60000	0.50000	1,5	0.01385	0.86801	0.11814	10,42	0.04337
0.60000	0.40000	2,2	0.00419	0.73252	0.26328	11,08	0.07070
0.60000	0.30000	3,5	0.00109	0.50799	0.49092	11,62	0.08889
0.60000	0.20000	6.0	0.00019	0.24846	0.75135	11,25	0.06971
0.60000	0.14000	9.3	0.00004	0.11792	0.88204	10,40	0.04045
0.60000	0.07000	20,0	0.00000	0.02818	0.97182	9 02	0.01011
0.60000	0.10000	13,5	0.00001	0.05972	0 94027	9,67	0.02174
0.70000	0.70000	1,0	0.04932	0.90135	0 04932	12,09	0.08025
0.70000	0.60000	1,6	0.01275	0.85129	0.13596	11,48	0.07332
0.70000	0.50000	2,3	0.00369	0.71030	0.28601	11,61	0.08925
0.70000	0.40000	3,5	0.00109	0.50799	0.49092	11,62	0.08889
0.70000	0.30000	5,4	0.00027	0.28985	0.70987	10,96	0.06176
0.70000	0.20000	9,3	0.00005	0.11760	0.88236	9,55	0.02519
0.70000	0.10000	20,0	0.00000	0.02791	0.97209	7,83	0.00413
0.70000	0.10000	21,0	0.00000	0.02541	0.97459	7,75	0.00363
0.80000	0.80000	1,0	0.06858	0.86284	0.06858	15,78	0.22288
0.80000	0.70000	1,7	0.01154	0.79729	0.19117	14,25	0.17222
0.80000	0.60000	2.7	0.00235	0.62853	0.36912	13,54	0.15923
0.80000	0.50000	4,0	0.00065	0.42926	0.57009	12,64	0.12312
0.80000	0.40000	6,0	0.00019	0.24846	0.75135	11,25	0.06971
0.80000	0.30000	9,3	0.00005	0.11760	0.88236	9,55	0.02519
0.80000	0.20000	16,0	0.00001	0.04221	0.95778	7,83	0.00461
0.80000	0.17000	20,0	0.00000	0.02727	0.97272	7,30	0.00208
0.80000	0.10000	36.0	0.00000	0.00855	0.99145	6,33	0.00023
0.90000	0.90000	1,0	0.10857	0.78285	0.10857	21,72	0.61937
0.90000	0.80000	2,2	0.01145	0.64912	0.33943	19,34	0.44272
0.90000	0.70000	3,9	0.00103	0.41986	0.57911	17,33	0.31627
0.90000	0.60000	6,0	0.00014	0.23957	0.76029	14,82	0.17829
0.90000	0.50000	9,0	0.00004	0.12490	0.87507	12,11	0.07488
0.90000	0.49000	9,3	0.00003	0.11723	0.88274	11,87	0.06806
0.90000	0.40000	13,5	0.00001	0.05972	0.94027	9,67	0.02174
0.90000	0.31000	20,0	0.00000	0.02797	0.97202	7,92	0.00452
0.90000	0.30000	21,0	0.00000	0.02541	0.97459	7,75	0.00363
0.90000	0.20000	36,0	0.00000	0.00855	0.99145	6,33	0.00023
0.90000	0.10000	81,0	0.00000	0.00154	0.99846	5,27	0.00000

Table 5.9
Comparison with Oksoy Plan 4

p_1	p_2	t	$P^*(H_a)$	$P_s(H_a)$	ASN*	ASN
.1	.1	1.00	0.0090	0.2172	11	21.70
.2	.2	1.00	0.0586	0.1372	12	15.78
.2	.1	2.25	0.0778	0.3509	13	19.34
.3	.3	1.00	0.1159	0.0987	11	12.09
.3	.1	3.85	0.3077	0.5801	14	17.33
.4	.4	1.00	0.1544	0.0881	11	10.39
.4	.1	6.00	0.6182	0.7604	13	14.82
.5	.5	1.00	0.1675	0.0857	11	9.93
.5	.1	9.00	0.8848	0.8751	11	12.11
.6	.6	1.00	0.1544	0.0881	11	10.39
.6	.2	6.00	0.8210	0.7515	10	11.25
.7	.7	1.00	0.1159	0.0987	11	12.09
.7	.2	9.33	0.9354	0.8824	8	9.55
.8	.8	1.00	0.0586	0.1372	12	15.78
.8	.3	9.33	0.9354	0.8824	8	12.11
.9	.9	1.00	0.0090	0.2172	11	21.70
.9	.5	9.00	0.8848	0.8751	11	12.11

the probabilities (e.g., .2 vs. .1 and .3 vs. 1.). The former, as explained in Section 4.1 is the preferred method of comparing unknown proportions.

The ASN functions of the two tests do not differ appreciably except when p_1 and p_2 are both small or both large, in which case the ASN of the new test increases considerably (as is expected), to correct for the lower average amount of information obtained per trial for such values of p_1 and p_2 . Similar comparisons can be made with the tests proposed by Armitage (1960), which are based on the same statistic, D=x-y. His tests, however, are truncated after a fixed number of untied pairs have been observed, causing the ASN to be extremely large for values of $p_1=p_2$ which are close to 0 or 1.

From these comparisons, it seems reasonable to conclude that the new tests given here have a decided advantage when using such sequential tests and when the sample sizes will generally be small. This will be especially true when extreme values of p_1 and p_2 can be expected, for which larger samples are necessary for the central limit theorem to become applicable, allowing the simpler statistic of Armitage and Oksoy to become acceptable for such tests. In any case, the new test will not be any worse than tests which do not use a sufficient statistic.

CHAPTER 6

ESTIMATING PARAMETERS OF A 2x2 CONTINGENCY TABLE AFTER A SEQUENTIAL TEST

6.0 INTRODUCTION

Often, after completion of sequential tests of hypotheses, it is desirable or necessary to estimate the parameters in question. This subject is treated here. The general method of estimation used here is due to Goss (1974a) and Schmee (1974). Some of the preliminaries for the material presented here, including a brief history of sequential estimation, an explanation of the general method of estimation given by Schmee and Goss, and a section describing the interpretation of these estimates, is contained in Meeker (1975) and will be referred to below. The first section of this chapter reviews the estimation of the binomial parameter, p, as treated by Goss (1974a). Section 6.2 applies the general method to estimation of the parameters of 2x2 contingency tables. The last section illustrates the procedures with a numerical example.

6.1 ESTIMATION IN THE BINOMIAL CASE

The following is a development of the posterior distribution and estimation procedures for the binomial distribution parameter p. For the cases considered here (although it is not true in general) the estimates will be independent of the stopping rule; that is,

the estimates (and confidence intervals) will depend only on the observed data at the termination of the test, and do not depend on the particular stopping rules (except that the stopping rules dictate where the sequential test may terminate).

The probability mass function of the binomial distribution is

$$b(x,n,p)=\binom{n}{x} p^x(1-p)^{n-x} \tag{6.1}$$

Following Goss (1974a) and the general procedure outlined in the preliminaries, the likelihood of a sample point (n,x) (i.e., of the observed data) at the termination of a sequential test is

$$b_S(x,p,n)=K(n,x)p^x(1-p)^{n-x} \tag{6.2}$$

where K(n,x) is the number of admissible paths from the origin (0,0) to the point (n,x). The posterior distribution of p (assuming a uniform (0-1) prior) is then

$$G(p,x,n)=\frac{b_S(x,p,n)}{\int_0^1 b_S(x,q,n)dq}=\frac{p^x(1-p)^{n-x}}{\int_0^1 q^x(1-q)^{n-x}dq} \tag{6.3}$$

From the posterior, one can find a point estimate of a parameter by using, for example, the mean of the distribution. The expected value of p with respect to the posterior distribution is

$$\hat{p}=E(p)=\int_0^1 p\cdot G(p,n,x)dp \tag{6.4}$$

The complete beta function is defined as

$$B(a,b)=\int_0^1 q^{a-1}(1-q)^{b-1}dq = (\Gamma(a)\Gamma(b))/\Gamma(a+b) \qquad (6.5)$$

where $\Gamma(\cdot)$ is the well-known gamma function and $\Gamma(k+1)=k!$ for integer k. From this (6.4) reduces to

$$\hat{p}= \frac{B(x+2,n-x+1)}{B(x+1,n-x+1)} = (x+1)/(n+2) \qquad (6.6)$$

Confidence intervals (with a Bayesian interpretation) can be constructed by finding values $\underset{\sim}{p}$ and $\tilde{p}$ such that

$$\int_{\underset{\sim}{p}}^{\tilde{p}} G(p,x,n)dp=1-\alpha \qquad (6.7)$$

giving a $100(1-\alpha)\%$ confidence level.

The upper and lower confidence limits $\tilde{p}$ and $\underset{\sim}{p}$ can be chosen in a number of different ways. If a one-sided interval is desired, $\tilde{p}=1$ or $\underset{\sim}{p}=0$ for a lower and upper tailed one-sided interval respectively. For a two-sided interval, the values can be chosen to minimize the interval length $\tilde{p}-\underset{\sim}{p}$ or to have equal probability $(\alpha/2)$ in each tail of the posterior distribution.

By using the incomplete beta distribution function,

$$I_p(a,b)=(1/B(a,b))\int_0^p q^{a-1}(1-q)^{b-1}dq \qquad (6.8)$$

(a very thorough treatment of this function is given, for example, by Abramowitz and Stegun (1965) and Johnson and Kotz (1971)), the values $\tilde{p}$ and $\underset{\sim}{p}$ are easily found. If

$$I_p(a,b)=\gamma$$

then (6.9)

$$p=I_\gamma^{-1}(a,b)$$

is the inverse beta distribution function and p is the $100\gamma^{th}$

percentile of the distribution. An equal tailed $100(1-\alpha)\%$ confidence interval can be found, for example, by

$$\underset{\sim}{p}=I^{-1}_{\alpha/2}(x+1,n-x+1)$$
$$\tilde{p}=I^{-1}_{1-\alpha/2}(x+1,n-x+1) \qquad (6.10)$$

6.2 ESTIMATION OF THE PARAMETERS OF A 2x2 CONTINGENCY TABLE

This section treats the estimation of the parameters of a 2x2 contingency table; the estimation is to be performed at the completion of a sequential test. The estimation procedure, a Bayesian approach, is based on the general method given by Goss (1974a) and Schmee (1974) and in particular, its application to the binomial distribution, as described in the previous section.

The underlying probability model of a 2x2 contingency table with both margins random (and the probability mass functions assumed to be unknown) is the multinomial distribution shown in (3.1). The observed data from such a 2x2 contingency table and the corresponding probabilities are shown in Figures 3.2 and 3.1 respectively.

The multinomial probability mass function in (3.1) can be factored as follows.

$$P_F(x,n_{1.},n_{.1},n)= \qquad (6.11)$$
$$b(n_{1.},p_{1.},n)b(x,p_1,n_{1.})b(n_{.1}-x,p_2,n-n_{1.})$$

where $b(x,p,n)=\binom{n}{x}p^{x}(1-p)^{n-x}$ is the binomial distribution

and

$$p_1=p_{11}/p_{1.}$$
$$p_2=(p_{.1}-p_{11})/(1-p_{1.}) \qquad (6.12)$$

are the conditional (on the right-hand margin ($n_{1.}$) of Figure 3.2) probabilities for each of the rows.

Because (3.1) factors exactly into the binomial distributions in (6.11), the three parameters p_1, p_2 and $p_{1.}$, which completely describe the state of nature, can be estimated independently. Also, p_1 and p_2 are often more important than the individual cell probabilities within the table, as their equality signifies independence of the row and column characteristics being observed. Lindley (1964) uses similar factorization for Beyesian analysis of general RxC contingency tables.

Using the above and the results presented in Section 6.1, one can find estimates and confidence intervals for these three parameters of a 2x2 contingency table. Estimates (i.e., the expected value with respect to the posterior distribution) for the three independent parameters in (6.11) are

$$\hat{p}_{1.}=(n_{1.}+1)/(n+2)$$
$$\hat{p}_1=(x+1)/(n_{1.}+2) \qquad (6.13)$$
$$\hat{p}_2=(n_{.1}-x+1)/(n-n_{1.}+2).$$

Using the inverse incomplete beta distribution as in (6.10), one can find (independent) Bayesian confidence intervals for each of these parameters. For example,

$$\underset{\sim}{p}_1 = I^{-1}_{\alpha/2}(x+1, n_{1.}-x+1) \qquad (6.14)$$
$$\tilde{p}_1 = I^{-1}_{1-\alpha/2}(x+1, n_{1.}-x+1)$$

give the upper and lower limits for a two-sided $100(1-\alpha)\%$ Bayesian confidence interval for p_1. This can be done in a similar manner for p_2 and $p_{1.}$.

Because of the independence of the three binomial distributions in (6.11), simultaneous confidence intervals for these parameters can easily be found. For example, in order to find a joint $100(1-\alpha)\%$ confidence region for all three of the parameters, one should choose the confidence level for each individual interval to be $100(1-\gamma_i)$, i=1,3 such that

$$\alpha = 1 = \prod_i (1-\gamma_i). \qquad (6.15)$$

The estimation procedure for the binomial distribution is easily generalized to treat the individual probabilities of the multinomial distribution of the 2x2 contingency tables considered here. The expected value (with respect to the posterior distribution) of the probability of a given cell of the multinomial distribution (assuming a uniform (0-1) prior distribution) can be shown (Lindley (1975) and Good (1965)) to reduce to

$$\hat{p} = (x+1)/(n+k) \qquad (6.16)$$

where x is the observed count in the cell being considered, n is

the sample size and k is the number of cells in the multinomial distribution. Equation (6.6), for the binomial distribution, can be seen to be a special case (i.e., k=2) of (6.16).

The above can be used to obtain point estimates of each of the cell probabilities. Letting Π_i, i=1,4 equal the individual probabilities of the cells shown in Figure 3.1, these estimates are

$$\begin{aligned}
\hat{\Pi}_1 &= (x+1)/(n+4) \\
\hat{\Pi}_2 &= (n_{1.}-x+1)/(n+4) \\
\hat{\Pi}_3 &= (n_{.1}-x+1)/(n+4) \\
\hat{\Pi}_4 &= (n-n_{1.}-n_{.1}+x+1)/(n+4)
\end{aligned} \tag{6.17}$$

Bayesian confidence intervals for the individual cell probabilities can be constructed in a manner analogous to that for the binomial distribution, treated in the previous section. For example, a 100(1-α)% upper confidence limit for Π_1 is

$$\tilde{\Pi}_1 = I^{-1}_{1-\alpha/2}(x+1, n-x+4) \tag{6.18}$$

Upper and lower confidence limits for the other parameters are similarly constructed. This is done as follows:

$$\begin{aligned}
\underset{\sim}{\Pi}_1 &= I^{-1}_{\alpha/2}(x+1, n-x+1) \\
\tilde{\Pi}_2 &= I^{-1}_{1-\alpha/2}(n_{1.}-x+1, n-n_{1.}+x+4) \\
\underset{\sim}{\Pi}_2 &= I^{-1}_{\alpha/2}(n_{1.}-x+1, n-n_{1.}+x+4)
\end{aligned} \tag{6.19}$$

$$\tilde{\pi}_3 = I^{-1}_{1-\alpha/2}(n_{.1}-x+1, n-n_{.1}+x+4)$$

$$\underset{\sim}{\pi}_3 = I^{-1}_{\alpha/2}(n_{.1}-x+1, n-n_{.1}+x+4)$$

$$\tilde{\pi}_4 = I^{-1}_{1-\alpha/2}(n-n_{1.}-n_{.1}+x+1, n_{1.}+n_{.1}-x+4)$$

$$\underset{\sim}{\pi}_4 = I^{-1}_{\alpha/2}(n-n_{1.}-n_{.1}+x+1, n_{1.}+n_{.1}-x+4)$$

6.3 NUMERICAL EXAMPLE OF THE ESTIMATION PROCEDURE

Suppose that a sequential test is terminated at trial 25 with the observed data shown in Figure 6.1.

x=8	2	$n_{1.}$=10
7	8	15
$n_{.1}$=15	10	n=25

Figure 6.1 Observed Data from a 2x2 Table

The estimates and 90% confidence intervals for the parameters of this example are shown in Table 6.1.

Table 6.1
Point Estimates and
90% Confidence Limits

	Estimate	Lower Limit	Upper Limit
p1	0.407	0.234	0.593
p1	0.750	0.484	0.939
p2	0.471	0.248	0.700
$\tilde{\pi}1$	0.310	0.197	0.409
$\underset{\sim}{\pi}2$	0.103	0.039	0.173
$\tilde{\pi}3$	0.276	0.168	0.372
$\underset{\sim}{\pi}4$	0.310	0.197	0.409

CHAPTER 7

CONCLUSION AND DISCUSSION OF POSSIBLE AREAS FOR FURTHER RESEARCH

7.0 INTRODUCTION

This chapter begins with a brief review of the models which can be formulated in terms of a 2x2 contingency table. This is followed by a discussion of some possible refinements and areas for further research, and some concluding remarks.

7.1 REVIEW OF 2x2 CONTINGENCY TABLE MODELS

The different probability models which can be formulated in terms of 2x2 tables are treated at some length in Sections 1.2 and 1.3. There, six different models are discussed. These models differ with respect to the number of margins which are "observable" (i.e., margins which can be controlled by the experimenter) and the number of marginal probability distributions which are known (knowledge of the probability function of a margin which is controlled by the experimenter, of course, has no additional value). These six models are depicted in Figure 7.1 and are explained below.

Case I This model is used when both marginal totals are random variables and marginal probability distributions are unknown. Inferences concerning the degree of dependence (measured by the cross product ratio) are made conditional on the

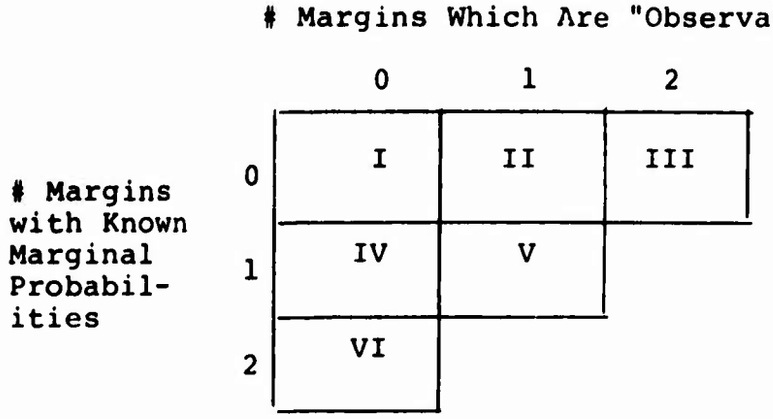

Figure 7.1 2x2 Contingency Table Models

observed values of the ancillary statistics, the marginal totals. This case is treated in Chapter 4.

Case II This model differs from that of Case I in that one of the marginal populations is "observable." That is, the experimenter can choose at will an observation from either category of the "observable" population. The test is then for the equality (or some degree of inequality) of two unknown binomial proportions. Inferences for this model are made by conditioning on the observed value of the ancillary statistic, which is the total number of successes for both populations. This model is treated in Chapter 5 for the special case when one observation is taken from each population at each stage of the test (a common sampling procedure). The method given there, however, is general and can be applied to other problems when one margin is controlled in some other prespecified manner.

Case III In this model, both marginal totals can be fixed in advance. This is not a commonly used model. The classic example of such a test is Fisher's tea-tasting experiment, briefly

mentioned in Section 1.3. Sequential applications for this model seem limited.

Case IV This model occurs when both marginal totals are random variables and the probability distribution of one of the totals is known. This case has not been treated here.

Case V When one marginal population is "observable" (i.e., can be controlled by the experimenter) and the other marginal distribution has a known probability distribtion, the problem can be reduced to a simple binomial model by taking all observations from one of the categories of the "observable" population, as explained in Section 1.2. Such a procedure gives an asymptotically most powerful test (Lehmann, 1959).

Case VI This model is similar to that of Case I except that in this case the marginal probability distributions are known. The null hypothesis of independence is most conveniently expressed as

$$H_0: \quad p_{11}=p_{1.}p_{.1} \tag{7.1}$$

There are no nuisance parameters in this model (p_{11} is the only unknown parameter) and an unconditional test of the hypothesis in (7.1) is easily found. Sequential methods for this case are treated in Chapter 3.

7.2 POSSIBLE AREAS FOR FURTHER RESEARCH

There are several topics related to the above results which might lead to further research. Some of these are briefly outlined below.

Evaluation of the exact properties of the sequential tests for 2x2 contingency tables which are given in Chapters 3 and 4 involves a large amount of computation when the test is not truncated at a relatively small sample size. This problem is less severe for the cases treated in Chapter 5. The numerical examples given in Chapters 3 and 4 were truncated at trial 25. Truncation at larger sample sizes was not feasible because of limitations of computer memory with available facilities. With a medium size computer (e.g., 32k words of memory), it would be possible to run cases up to trial 100. For tests requiring sample sizes which are much larger than this, some other methods might be developed. Asymptotic theory might be of some assistance here.

It is well known that the χ^2 distribution can be used to approximate the multinomial distribution associated with a contingency table and can therefore be used to make tests of significance for this model. The observed marginal totals are used to estimate the marginal probabilities if they are unknown. The χ^2 approximation is valid when the expected values of each of the contingency table's cells is of sufficient size (usually an expected value of 5 is specified, although some argue for a lower value).

The usual χ^2 statistic for a general RxC contingency table is computed as

$$\chi^2 = \sum_{i=1}^{R} \sum_{j=1}^{C} (n_{ij}-E_{ij})^2/E_{ij} \qquad (7.2)$$

where $E_{ij}=n_{i.}n_{.j}/n$ is the expected value (under the null hypothesis of independence) of cell (i,j) and

$$n=\sum_{i=1}^{R}\sum_{j=1}^{C} n_{ij}=\sum_{i=1}^{R} n_{i.}=\sum_{j=1}^{C} n_{.j} \qquad (7.3)$$

As mentioned in Section 1.3, a half integer continuity correction can also be used here. The χ^2 statistic has one degree of freedom for cases I, II and III; two degrees of freedom for cases IV and V; and three degrees of freedom for case VI, as shown in Figure 7.1.

Harkness (1959) treats the asymptotic power of such tests. He shows that for non-independent 2x2 tables, the χ^2 statistic in (7.1) asymptotically follows a non-central χ^2 distribution with non-centrality parameter

$$\delta = \frac{np_{1.}p_{.1}(1-p_{11}/(p_{1.}p_{.1}))^2}{((1-p_{1.})(1-p_{.1}))} \qquad (7.4)$$

(which is zero under the null hypothesis of independence).

For sequential analysis of 2x2 contingency tables when large sample sizes are required, it would be reasonable to have a test procedure based on a model similar to this. Such a test might be based on the non-centrality parameter (corrected for the sample size), using some function of the χ^2 statistic in (7.1) (corrected for the sample size as a test statistic). A reasonable statistic might be the phi coefficient

$$\phi=(\chi^2/n)^{\frac{1}{2}}. \qquad (7.5)$$

Goodman and Kruskal (1954) give an account of this statistic as

a measure of association. Such a test could be evaluated using the methods given in Chapter 3, still leaving the computational difficulties for large samples. Approximate properties of such tests could be obtained by using the direct method to evaluate the non-central χ^2 distribution under sequential test rules (such a procedure has not yet been investigated) or with Monte Carlo techniques. Such a test, based on the χ^2 statistic (or some function of it), has two advantages:

1. The test statistic has only one dimension.
2. Results could be easily extended to general RxC or multidimensional contingency tables.

At the beginning of the sequential test, when the χ^2 distribution is not applicable (say for $n<25$), special considerations must be used. Two possibilities seem reasonable:

1. Use the small sample procedures given above for $n<25$.
2. Do not allow termination of the sequential test until $n=25$, or until the expected values of each of the cells are "large enough."

The development of a test procedure similar to the above would be valuable in situations where relatively large samples will be required to obtain the desired test properties.

Finally, it should be pointed out that the results in Chapters 3, 4 and 5 remain valid for large samples (although the procedures are more difficult than for asymptotic (e.g., χ^2) tests). Also, it is not too difficult (given a small computer

program like the one listed in the Appendix) to compute the individual critical values for the two and three dimensional test statistics for such large sample sizes. Tables for complete plans, however, would be quite lengthy. The difficult problem is in finding the exact test properties for tests requiring such large samples.

One problem which has arisen with the use of sequential analysis is that in the past exact test properties of the sequential tests were unknown. Researchers and experimenters usually had to rely on the sometimes crude approximations and bounds given by Wald (1947) and others. For cases where no such approximations are available, Monte Carlo techniques have been used (and sometimes misused). The direct method of sequential analysis has provided a vehicle for overcoming this problem. The results obtained thus far with the direct method of sequential analysis (see the references given in Section 2.2) have been substantial and have shown that the Wald regions give generally good results, even when the sequential test is terminated at or near the sample size necessary for a comparable fixed size sample test. When a given test procedure does not have the desired test properties, the direct method can be used to evaluate other alternate regions.

Schmee (1974) uses the direct method in the presence of a nuisance parameter. In his treatment of the sequential t-test, σ, the standard deviation of the normal population, is a nuisance parameter. As explained in Chapters 4 and 5, the marginal probabilities of 2x2 contingency tables are nuisance parameters when

one wishes to make inferences concerning the degree of dependence (i.e., inferences concerning t, the cross product or odds ratio). The problem in the present case, however, is somewhat different than that of the sequential t-test. In the sequential t-test, the parameter under test is $d=\mu/\sigma$, where μ is the mean of the normal population. The test properties for the sequential t-test will be exactly the same for equal values of d, irrespective of the actual values of μ and σ. This is the property of invariance (Hall, Wijsman and Ghosh, 1965). This is not true in the case of the cross product (or odds) ratio with respect to nuisance parameter(s) of a 2x2 contingency table. That is, the test properties will vary over the parameter space for equal values of the cross product (or odds) ratio. It is encouraging to note, however, that the test properties do not vary appreciably when the probabilities of continuation for a given point in the parameter space, is small, as shown in the numerical examples of Chapter 5. This is discussed below.

However, for certain values of the nuisance parameters, the test properties will deviate considerably from the specified "desired" error probabilities. It is sometimes a problem to attain the desired error probabilities in the presence of such nuisance parameters, but several methods of approach (all relying heavily on the direct method for test property evaluation) are suggested.

First, one can modify the critical values (usually, but not necessarily at the truncation point, n_0) to favor one hypothesis or the other. This can be done in a systematic manner by examination of the probabilities of reaching the points in question

(these probabilities are available from the direct method). Some rules of thumb can be devised for this procedure by examination of the distribution of the ancillary statistics under states of nature where the test properties need to be changed. For example, if one desires to change (for the most part) the probability of acceptance of one hypothesis or another in a test for the equality of two unknown binomial proportions (as treated in Chapter 5), for values of $p_1=p_2$ when both are small, one would modify the truncation rule for small values of the ancillary statistic $n_{.1}$, the total number of successes from both populations. Such a modification will have very little effect on the test properties for values of $p_1=p_2$ near 0.5. This procedure is easily generalized for the 2x2 tables treated in Chapter 4. Also, such modification at trial n_0 will have the largest effect on those points in the parameter space which have the largest probability of continuation at trial n_0-1. It must be remembered that such modification will result, for example, in a reduction of the α error probabilities, with a resulting loss in power at alternates to the null hypothesis; the hope being that the relative gain will exceed this loss. Use of the direct method will facilitate such modifications. It should be noted that modification of the region in this manner will not affect the distribution of the DSN or the ASN function.

When the probability of continuation at trial n_0-1 is small at points in the parameter space where the error probabilities deviate from the desired values, another approach may have to be

used (this approach may be the best in any case, however). This is because the above method of region modification will have little or no effect on the error probabilities.

The numerical example given in Table 5.8 shows that for most points in the (p_1,p_2) parameter space, the actual β error probabilities are considerably less than the specified values (the α error probabilities, for most points, are near the desired values). For most of these points, the probability of continuation at trial n_0-1 is small. This indicates that it would be reasonable to change the values with which the likelihood ratio is compared at each trial (or equivalently, the specified "desired" error probabilities) in order to achieve the desired test properties. In the present example, one would allow an increase in the β error probabilities in the hope of making some gains with respect to the ASN function. Table 7.1 shows the test properties for the same example, except that the desired error probabilities were specified as $\alpha_1=\alpha_2=0.024$ and $\beta_1=\beta_2=0.45$. The resulting test properties after this change are closer to the original desired values and the ASN function has decreased somewhat. This procedure could be repeated until the desired error probabilities are more closely approached.

For certain cases, especially for extreme values of the nuisance parameters (e.g., when it is necessary to discriminate between p_1 and p_2 and both are expected to be small), the above regions modification procedures may not be able to give satisfactory results. In such cases, it will be necessary to increase the

Table 7.1
Three Decision $p_1=p_2$
Example Test Properties

P_1	P_2	t	$P(H_1)$	$P(H_0)$	$P(H_2)$	ASN	$P(C_{n_0-1})$
0.10000	0.10000	1,0	0.06929	0.86143	0.06929	16,31	0.33713
0.10000	0.01000	9,3	0.00249	0.43891	0.55860	23,10	0.83099
0.10000	0.01000	20,0	0.00060	0.35594	0.64345	23,99	0.90213
0.20000	0.20000	1,0	0.03587	0.92827	0.03587	11,67	0.11396
0.20000	0.10000	2,2	0.00538	0.78090	0.21372	14,93	0.28148
0.20000	0.03000	9,3	0.00053	0.32107	0.67841	20,20	0.55993
0.20000	0.01000	20,0	0.00014	0.17677	0.82309	21,41	0.61114
0.30000	0.30000	1,0	0.02838	0.94325	0.02838	9,65	0.04162
0.30000	0.20000	1,7	0.00552	0.87365	0.12083	11,18	0.10159
0.30000	0.10000	3.9	0.00039	0.57722	0.42239	14,60	0.24689
0.30000	0.04000	9,3	0.00005	0.26965	0.73031	16,75	0.29142
0.30000	0.02000	20,0	0.00001	0.12729	0.87270	17,40	0.27217
0.40000	0.40000	1,0	0.02544	0.94913	0.02544	8,82	0.02084
0.40000	0.30000	1,6	0.00623	0.90352	0.09025	9,64	0.04460
0.40000	0.20000	2,7	0.00094	0.72900	0.27006	11,55	0.11121
0.40000	0.10000	6,0	0.00004	0.37472	0.62524	13,61	0.16001
0.40000	0.07000	9,3	0.00001	0.23343	0.76656	13,92	0.14241
0.40000	0.03000	20,0	0.00000	0.09886	0.90114	13,83	0.09771
0.50000	0.50000	1,0	0.02456	0.95088	0.02456	8,58	0.01746
0.50000	0.40000	1,5	0.00646	0.91435	0.07919	9,07	0.03099
0.50000	0.30000	2,3	0.00136	0.78504	0.21360	10,36	0.06739
0.50000	0.20000	4,0	0.00018	0.53489	0.46493	11,76	0.09986
0.50000	0.10000	9,0	0.00001	0.21696	0.78303	12,05	0.07447
0.50000	0.10000	9,3	0.00001	0.20737	0.79263	12,03	0.07233
0.50000	0.05000	20,0	0.00000	0.07984	0.92016	11,38	0.03406
0.60000	0.60000	1,0	0.02544	0.94913	0.02544	8,82	0.02084
0.60000	0.50000	1,5	0.00646	0.91435	0.07919	9,07	0.03099
0.60000	0.40000	2,2	0.00149	0.79977	0.19874	10,03	0.05946
0.60000	0.30000	3,5	0.00028	0.59601	0.40371	11,06	0.08175
0.60000	0.20000	6,0	0.00003	0.33647	0.66349	11,36	0.06791
0.60000	0.14000	9,3	0.00001	0.18761	0.81238	10,91	0.04139
0.60000	0.07000	20,0	0.00000	0.06487	0.93513	9,84	0.01237
0.60000	0.10000	13,5	0.00000	0.11195	0.88805	10,37	0.02383
0.70000	0.70000	1.0	0.02838	0.94325	0.02838	9,65	0.04162
0.70000	0.60000	1,6	0.00623	0.90352	0.09025	9,64	0.04460
0.70000	0.50000	2,3	0.00136	0.78504	0.21360	10,36	0.06739
0.70000	0.40000	3,5	0.00028	0.59601	0.40371	11,06	0.08175
0.70000	0.30000	5,4	0.00005	0.37262	0.62733	11,13	0.06730
0.70000	0.20000	9,3	0.00001	0.17621	0.82378	10,31	0.03100
0.70000	0.10000	20,0	0.00000	0.05436	0.94564	8,90	0.00535
0.70000	0.10000	21,0	0.00000	0.05042	0.94957	8,82	0.00472
0.80000	0.80000	1,0	0.03587	0.92827	0.03587	11,67	0.11396
0.80000	0.70000	1,7	0.00552	0.87365	0.12083	11,18	0.10159
0.80000	0.60000	2,7	0.00094	0.72900	0.27006	11,55	0.11121
0.80000	0.50000	4,0	0.00018	0.53489	0.46493	11,76	0.09986
0.80000	0.40000	6,0	0.00003	0.33647	0.66349	11,36	0.06791
0.80000	0.30000	9,3	0.00001	0.17621	0.82378	10,31	0.03100
0.80000	0.20000	16,0	0.00000	0.07269	0.92731	8,90	0.00729
0.80000	0.17000	20,0	0.00000	0.04986	0.95014	8,42	0.00354
0.80000	0.10000	36,0	0.00000	0.01840	0.98160	7,49	0.00043
0.90000	0.90000	1,0	0.06929	0.86143	0.06929	16,31	0.33713
0.90000	0.80000	2,2	0.00538	0.78090	0.21372	14,93	0.28148
0.90000	0.70000	3,9	0.00039	0.57722	0.42239	14,60	0.24689
0.90000	0.60000	6,0	0.00004	0.37472	0.62524	13,61	0.16001
0.90000	0.50000	9,0	0.00001	0.21696	0.78303	12,05	0.07447
0.90000	0.49000	9.3	0.00001	0.20529	0.79470	11,90	0.06826
0.90000	0.40000	13,5	0.00000	0.11195	0.88805	10,37	0.02383
0.90000	0.31000	20,0	0.00000	0.05519	0.94481	8,97	0.00574
0.90000	0.30000	21,0	0.00000	0.05042	0.94957	8,82	0.00472
0.90000	0.20000	36,0	0.00000	0.01840	0.98160	7,49	0.00043
0.90000	0.10000	81,0	0.00000	0.00405	0.99595	6,40	0.00001

the value of n_0, the truncation point of the sequential test.

One other important and desirable characteristic of the sequential tests with nuisance parameters presented here is that they give a method of obtaining a test procedure which affords approximately equal error protection against specified (by H_1 or H_2) values of alternate hypotheses (the hypotheses being specified in terms of the cross product (or odds) ratio), with a certain amount of "invariance" to the actual values of the nuisance parameter(s). For example, as shown in Table 5.7, the fixed size (UMPUT) (see 4.10) for the hypotheses in (5.12) has a probability of rejecting H_0 equal to 0.05 for all $p_1=p_2$. For equal values of t other than one in the (p_1,p_2) parameter space (see Figure 4.1), however, the power varies considerably with the actual values of the nuisance parameters.

Because the amount of information obtained from a given sample depends on the observed values of the ancillary statistic(s) (whose distribution depends only on the values of the nuisance parameter(s)), there is no fixed size test procedure which will give even approximately equal protection (with respect to the power function) along contours of equal values of $t\neq 1$ in the (p_1,p_2) parameter space. The sequential procedures presented here help correct for this and the power function of such tests will be relatively constant over such contours in that part of the (p_1,p_2) space where the probability of continuation to the truncation trial n_0 $(P(C_{n_0-1}))$ is small. This is evidenced, for example, in the test properties of the numerical examples which are given in Tables 5.6, 5.8, and 7.1.

This "invariance" property (which is not invariance in the strict sense of Hall, Wijsman and Ghosh (1965), but is similar in nature) is present in the sequential tests for the 2x2 contingency tables treated in Chapter 4 as well as for those treated in Chapter 5.

Another area for possible further research is the application of some of the above results with respect to the wide variety of other statistical problems which can be formulated in terms of a 2x2 contingency table. Some of these models include the study of matched proportions in crossover designs (Gart, 1969) and the study of Poisson distributed incidence rates (Gart, 1974). Also, nonparametric two-sample tests for the equality of medians can be formulated in terms of a 2x2 contingency table (Owen, 1962). The methods presented here might be used in two ways to help solve these problems sequentially. That is,

1. To obtain a SPRT which is conditional on observed ancillary statistic(s), yielding a test based on a sufficient statistic (which is usually desirable).
2. To find the exact test properties of such tests, based on evaluation procedures similar to those given here.

The underlying probability distributions of these models are usually somewhat more complex (especially under alternatives to the null hypothesis) than the multinomial and binomial distributions considered here.

7.3 CONCLUSION

It is hoped that the results presented here will be valuable both in a practical sense and as a stimulus toward further investigation of sequential methods for related problems. Some new methods of finding sequential test regions have been investigated here. In addition, methods of exact evaluation of the properties for these (and other similar) tests have been developed, enabling the experimenter using such methods to know more precisely the size of the risks associated with a given test procedure, and to help one find the best test procedure for the problem at hand.

The results presented here should have wide applicability in situations where it is difficult or expensive to obtain observations or when data are naturally obtained sequentially. As shown in the numerical comparisons given in Chapters 3, 4 and 5, the sequential tests presented here are clearly superior (with respect to expected sample size requirements) to similar fixed size and other proposed sequential (in the case of the comparison of two unknown binomial proportions) procedures, allowing significant savings with respect to the necessary time and/or expense associated with sampling.

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Appendix

Computer Programs Used to Develop and Evaluate the Test Plans

```
1000  C
1010  C*************************************************************
1020  C*
1030  C*        THIS PROGRAM COMPUTES THE SEQUENTIAL TEST REGION
1040  C*        FOR TESTING THE INDEPENDENCE OF A 2X2 CONTINGENCY
1050  C*        TABLE WHEN THE MARGINAL PROBABILITIES ARE UNKNOWN.
1060  C*
1070  C*************************************************************
1080  C
1090        DIMENSION KT(101)
1100        DIMENSION KL(100),KU(100)
1110        COMMON IL,IU
1120        DATA INPUT,IOUT/50,66/
1130        IUP=0
1140        READ(INPUT,1212)IREG
1150  1212  FORMAT(I2)
1160        DO 25 I=1,101
1170  25    KT(I)=I-1
1180        READ(INPUT,62)ALPHA,BETA
1190        READ(INPUT,62)T2,T0
1200        AT0=ALOG(T0)
1210        AT2=ALOG(T2)
1220        WRITE(IOUT,233)T0,T2,ALPHA,BETA
1230  233   FORMAT("1T0=",F7.3/" T1=",F7.3/" ALPHA=",F7.3/"BETA=",F7.3///)
1240        READ(INPUT,62)XM0
1250        M0=XM0
1260  62    FORMAT(8F10.0)
1270        AL1=ALOG(BETA/(1.-ALPHA))
1280        BL1=ALOG((1.-BETA)/ALPHA)
1290        DO 22 N=1,M0
1300        NP1=N+1
1310        WRITE(IOUT,888)
1320  888   FORMAT("0")
1330        WRITE(IOUT,42)N
1340  42    FORMAT(" TRIAL ",I4,5X,"N.1")
1350        IF(N.NE.M0) GO TO 9
1360        AL1=(AL1+BL1)/2.
1370        BL1=AL1
1380        IF(IREG.LE.-1)IUP=-1
1390        IF(IREG.GE.1)IUP=1
1400        IF(IUP.EQ. 1)WRITE(IOUT,701)
1410  701   FORMAT(" REGION MOVE UP")
1420        IF(IUP.EQ.-1)WRITE(IOUT,700)
1430  700   FORMAT(" REGION MOVE DOWN")
1440  9     CONTINUE
1450        XN=N
```

```
      WRITE(IOUT,662)
662   FORMAT(3X,"N1.")
      WRITE(IOUT,49)(KT(I),I=1,NP1)
49    FORMAT(7X,15(I4,3X))
      DO 33 I=1,NP1
      IM1=I-1
      N1DOT=I-1
      XN1DOT=I-1
      DO 44 J=1,NP1
      NDOT1=J-1
      XNDOT1=J-1
      IU=MINO(NDOT1,N1DOT)+1
      IL=MAXO(NDOT1+N1DOT-N,0)+1
      FT0=FNOD(NDOT1,N1DOT,N,AT0)
      FT2=FNOD(NDOT1,N1DOT,N,AT2)
      KL(J)=ILNT((AL1+FT0-FT2)/(AT2-AT0))
      KU(J)=ILNT((BL1+FT0-FT2)/(AT2-AT0))    +1
      IF(KU(J).LT.-1)KL(J)=-1
44    CONTINUE
      WRITE(IOUT,41)IM1
41    FORMAT("+",I4)
      WRITE(IOUT,40)(KL(J),KU(J),J=1,NP1)
40    FORMAT(" ",4X,2X,15(1X,I2,",",I2,1X))
33    CONTINUE
22    CONTINUE
      STOP
      END
C
C*************************************************************
C*
C*         THIS PROGRAM COMPUTES THE SEQUENTIAL TEST REGION
C*         FOR TESTING THE EQUALITY OF TWO BINOMIAL PROPORTIONS
C*         WHEN ONE ITEM IS SELECTED FROM EACH POPULATION AT
C*         EACH TRIAL.
C*
C*************************************************************
C
      DIMENSION KT(101)
      DIMENSION KL(100),KU(100)
      COMMON IL,IU,G(4),X(4)
      DATA INPUT,IOUT/50,66/
      IUP=0
      READ(INPUT,1212)IREG
1212  FORMAT(I2)
      DO 25 I=1,101
25    KT(I)=I-1
      READ(INPUT,62)ALPHA,BETA
      READ(INPUT,62)T0,T1
      WRITE(IOUT,64)T0,T1,ALPHA,BETA
64    FORMAT("1T0=",F8.2/" T1=",F8.2/" ALPHA=",F8.3/" BETA=",F8.3///)
      AT0=ALOG(T0)
```

```
1970         AT1=ALOG(T1)
1980         READ(INPUT,62)XMO
1990         MO=XMO
2000 62      FORMAT(8F10.0)
2010         NP1=2*MO+1
2020         WRITE(IOUT,663)
2030 663     FORMAT(50X,"N.1")
2040         WRITE(IOUT,49)(KT(I),I=1,NP1)
2050         WRITE(IOUT,662)
2060 662     FORMAT(" TRIAL"/"  N1.")
2070         BL1=ALOG((1.-BETA)/ALPHA)
2080         AL1=ALOG(BETA/(1.-ALPHA))
2090         DO 22 N=  .0
2100         NP1=2*N.
2110         IF(N.NE.MO) GO TO 9
2120         AL1=(AL1+BL1)/2.
2130         BL1=AL1
2140         IF(IREG.GE.1)IUP=1
2150         IF(IREG.LE.-1)IUP=-1
2160         IF(IUP.NE.0)WRITE(IOUT,700)
2170 700     FORMAT(" REGION MOVE")
2180 9       CONTINUE
2190 49      FORMAT(7X,15(I4,3X))
2200          N1DOT=N
2210         DO 44 J=1,NP1
2220          NDOT1=J-1
2230         IL=MAX0(NDOT1-N,0)+1
2240         IU=MIN0(NDOT1,N)+1
2250         FTO=FNOD(NDOT1,N1DOT,2*N1DOT,ATO)
2260         FT1=FNOD(NDOT1,N1DOT,2*N1DOT,AT1)
2270         KL(J)=ILNT((AL1+FTO-FT1)/(AT1-ATO))
2280        @+IUP
2290         KU(J)=ILNT((BL1+FTO-FT1)/(AT1-ATO))+1
2300        @+IUP
2310         IF(KU(J).LT.-1)KL(J)=-1
2320 44      CONTINUE
2330         WRITE(IOUT,41)N
2340 41      FORMAT("+",I4)
2350         WRITE(IOUT,40)(KL(J),KU(J),J=1,NP1)
2360 40      FORMAT(" ",4X,2X,15(1X,I2,",",I2,1X))
2370 22      CONTINUE
2380         STOP
2390         END
2400 C
2410 C*****************************************************
2420 C*
2430 C*         THIS PROGRAM COMPUTES THE SEQUENTIAL TEST REGION
2440 C*         FOR TESTING THE INDEPENDENCE OF A 2X2 CONTINGENCY
2450 C*         TABLE WHEN THE MARGINAL PROBABILITIES ARE KNOWN.
2460 C*
2470 C*****************************************************
2480 C
```

```
2490        DIMENSION KT(101)
2500        DIMENSION KL(100),KU(100)
2510        DATA INPUT,IOUT/50,66/
2520        IUP=0
2530        READ(INPUT,1212)IREG
2540 1212   FORMAT(I2)
2550        DO 25 I=1,101
2560 25     KT(I)=I-1
2570        READ(INPUT,62)ALPHA,BETA
2580        READ(INPUT,62)P1DOT,PDOT1,P0,P1
2590        READ(INPUT,62)XM0
2600        WRITE(IOUT,67)P1DOT,PDOT1,P0,P1,ALPHA,BETA
2610 67     FORMAT("1P1.=",F7.3/" P.1=",F7.3/" P0= ",F7.3/" P1=",F7.3/
2620       @" ALPHA=",F7.3/" BETA=",F7.3///)
2630        M0=XM0
2640 62     FORMAT(8F10.0)
2650        XU1=ALOG(P1/P0)
2660        XU2=ALOG((PDOT1-P1)/(PDOT1-P1))
2670        XU3=ALOG((P1DOT-P1)/(P1DOT-P0))
2680        XU4=ALOG((1.-P1DOT-PDOT1+P0)/(1.-P1DOT-PDOT1+P1))
2690        ZU=XU1-XU2-XU3+XU4
2700        XU43=XU4-XU3
2710        XU42=XU4-XU2
2720        BL1=ALOG((1.-BETA)/ALPHA)
2730        AL1=ALOG(BETA/(1.-ALPHA))
2740        DO 22 N=1,M0
2750        NP1=N+1
2760        WRITE(IOUT,888)
2770 888    FORMAT("0")
2780        WRITE(IOUT,42)N
2790 42     FORMAT(" TRIAL ",I4,5X,"N.1")
2800        IF(N.NE.M0) GO TO 9
2810        AL1=(AL1+BL1)/2.
2820        BL1=AL1
2830        IF(IREG.LE.-1)IUP=-1
2840        IF(IREG.GE.1)IUP=1
2850        IF(IUP.EQ. 1)WRITE(IOUT,701)
2860 701    FORMAT(" REGION MOVE UP")
2870        IF(IUP.EQ.-1)WRITE(IOUT,700)
2880 700    FORMAT(" REGION MOVE DOWN")
2890 9      CONTINUE
2900        XN=N
2910        WRITE(IOUT,662)
2920 662    FORMAT(3X,"N1.")
2930        WRITE(IOUT,49)(KT(I),I=1,NP1)
2940 49     FORMAT(7X,15(I4,
2950        DO 33 I=1,NP1
2960        IM1=I-1
2970        XN1DOT=I-1
2980        DO 44 J=1,NP1
```

```
2990        XNDOT1=J-1
3000        XNU=XNDOT1*XU43+XN1DOT*XU42-XN*XU4
3010        KL(J)=ILNT((AL1+XNU)/ZU)
3020       @+IUP
3030        KU(J)=ILNT((BL1+XNU)/ZU)+1
3040       @+IUP
3050        IF(KL(J).LT.-1)KL(J)=-2
3060        IF(KU(J).LT.-1)KU(J)=-1
3070 44     CONTINUE
3080        WRITE(IOUT,41)IM1
3090 41     FORMAT(*+*,I4)
3100        WRITE(IOUT,40)(KL(J),KU(J),J=1,NP1)
3110 40     FORMAT(" ",4X,2X,15(1X,I2,",",I2,1X))
3120 33     CONTINUE
3130 22     CONTINUE
3140        STOP
3150        END
3160 C
3170 C*****************************************************
3180 C*
3190 C*       THIS PROGRAM FIGURES AND EVALUATES REGIONS FOR A SEQUENTIAL
3200 C*   TEST OF A 2X2 CONTINGENCY TABLE.  THE TEST IS BASED ON THE CROSS
3210 C*   PRODUCT RATIO.  TRUNCATION OF REGIONS IS ALLOWED.
3220 C*   SAMPLES ARE TO BE TAKEN IN PAIRS FROM TWO DIFFERENT POPULATION.
3230 C*   THIS PROGRAM IS FOR A THREE DECISION TEST.
3240 C*
3250 C*
3260 C*                  WILLIAM O. MEEKER, JR.
3270 C*                  INSTITUTE OF ADMINISTRATION AND MANAGEMENT
3280 C*                  UNION COLLEGE
3290 C*                  SCHENECTADY, NEW YORK 12308
3300 C*                  AUGUST 1974
3310 C*
3320 C*****************************************************************************
3330 C
3340        DIMENSION A(27,27),B(27,27)
3350        DIMENSION                ACC0(75),ACC1(75),ACC2(75)
3360        DIMENSION ACC0T(75),ACC1T(75),ACC2T(75),N9(75),PCH(75)
3370        DIMENSION PAB(75),PB(75),PA(75),PN(75)
3380 C
3390        DIMENSION X(4)
3400        INTEGER TWON
3410        REAL N9,NT9
3420        LOGICAL UP1,UP2,DOWN1,DOWN2
3430        COMMON IL,IU,G(4), X1,X2,X3,X4
3440        EQUIVALENCE           (X1,X(1)    )
3450        EQUIVALENCE(II,K)
3460        DATA IPNOUT/58/
3470        DATA INPUT,IOUT/50,66/
3480 C
3490 C         SPECIFY M1(LOWER), M0(MIDDLE), M2(UPPER)
```

```
3500 C          TAKE LOGS FOR LATER USE
3510 C
3520        READ(INPUT,1212)IPUN,IREG
3530 1212   FORMAT(2I1)
3540        IUP=0
3550        READ(INPUT,70)ALPHA1,BETA1,ALPHA2,BETA2,XM0,T2,T0,T1
3560        IF(T0.EQ.0.0)T0=1.0
3570        IF(T1.EQ.0.0)T1=1./T2
3580 70     FORMAT(8F10.0)
3590        M0=XM0
3600        AT1=ALOG(T1)
3610        AT0=ALOG(T0)
3620        AT2=ALOG(T2)
3630 C
3640 C          SET DESIRED ERROR PROBABILITIES, CRITICAL LIMITS AND THEIR LOG
3650        A1=ALPHA1/(1.-BETA1)
3660        B1=(1.-ALPHA1)/BETA1
3670        A2=BETA2/(1.-ALPHA2)
3680        B2=(1.-BETA2)/ALPHA2
3690        WRITE(IOUT,47)
3700 47     FORMAT(1H1,9X,1HT,7X,7HALOG(T)/)
3710        WRITE(IOUT,41)T1,AT1,T0,AT0,T2,AT2
3720 41     FORMAT(3H T1,2X,2(3X,F7.4)/3H T0,2X,2(3X,F7.4)/3H T2,2X,2(3X,F7.4)
3730       6)
3740        WRITE(IOUT,45)ALPHA1,BETA1
3750 45     FORMAT(///10H ALPHA1 = ,F5.3/9H BETA1 = ,F5.3      )
3760        WRITE(IOUT,945)ALPHA2,BETA2
3770 945    FORMAT(/  10H ALPHA2 = ,F5.3/9H BETA2 = ,F5.3      )
3780 C
3790 C          READ SELECTED ALTERNATE HYPOTHESES WHERE THE REGION IS TO BE E
3800 C
3810        I=0
3820 1      CONTINUE
3830        I=I+1
3840        READ(INPUT,70)P1,P2
3850        IF(P1.EQ.0.0) GO TO 9922
3860        Q2=1.-P2
3870        Q1=1.-P1
3880        PAB(I)=ALOG(   P2)
3890        PA(I)=ALOG(P1    )
3900        PB(I)=ALOG(   Q1)
3910        PN(I)=ALOG(   Q2)
3920        GO TO 1
3930 9922   NALT=I-1
3940        AL1=ALOG(A1)
3950        BL1=ALOG(B1)
3960        AL2=ALOG(A2)
3970        BL2=ALOG(B2)
3980 C
3990 C          INITILIZE
4000 C
```

```
4010          A(1,1)=.25
4020          A(1,2)=.25
4030          A(2,1)=.25
4040          A(2,2)=.25
4050          DO 4488 I=1,NALT
4060          ACC1T(I)=0.0
4070          ACC2T(I)=0.0
4080          ACC0T(I)=0.0
4090          N9(I)=0.0
4100 4488     CONTINUE
4110 C
4120 C             INCREMENT TRIAL NUMBER
4130 C
4140          DO 34 N=1,M0
4150                              WRITE(IOUT,66)N
4160 66       FORMAT(" TRIAL NUMBER ",I5)
4170          TWON=2*N
4180          N1=TWON+1
4190          I3=N+2
4200          DO 77 I=1,I3
4210          DO 77 J=1,I3
4220 77       B(I,J)=0.
4230          DO 4499 I=1,NALT
4240          ACC0 (I)=0.0
4250          ACC1 (I)=0.0
4260          ACC2 (I)=0.0
4270 4499     CONTINUE
4280          IF(N.NE.M0) GO TO 56
4290 C
4300 C             ALLOW TRUNCATION IF DESIRED
4310 C
4320          DO 4477 I=1,NALT
4330 4477     PCH(I)=1.-ACC0T(I)-ACC1T(I)-ACC2T(I)
4340           AL1=(AL1+BL1)/2.
4350          BL1=AL1
4360          AL2=(AL2+BL2)/2.
4370          BL2=AL2
4380          IF(IREG.EQ.1)IUP=1
4390          IF(IUP.EQ.1)WRITE(IOUT,700)
4400 700      FORMAT(" REGION MOVE")
4410 56       CONTINUE
4420          Q=FLOAT(N)*(0.69314718)
4430          Q=-2*Q
4440 C
4450 C             ENUMERATE ALL POSSIBLE BOTTOM MARGINS
4460 C
4470          DO 22 J=1,N1
4480          NDOT1=J-1
4490          NDOT2=TWON-NDOT1
4500 C
4510 C             FIGURE LOWER AND UPPER LIMITS
```

```
4520 C
4530       IU=MINO(NDOT1,N)+1
4540       IL=MAXO(NDOT1-N,0)+1
4550 C
4560 C          SKIP IF REGIONS ARE ALREADY SAVED
4570 C
4580       FT0=FNOD(NDOT1,N,TWON,AT0)
4590       FT1=FNOD(NDOT1,N,TWON,AT1)
4600       FT2=FNOD(NDOT1,N,TWON,AT2)
4610 C
4620 C          FIGURE CRITICAL VALUES OF REGIONS
4630 C
4640       KL1=ILNT((AL1+FT1-FT0)/(AT0-AT1))
4650      *-IUP
4660       KU1=ILNT((BL1+FT1-FT0)/(AT0-AT1))+1
4670      *-IUP
4680       KL2=ILNT((AL2+FT0-FT2)/(AT2-AT0))
4690      *+IUP
4700       KU2=ILNT((BL2+FT0-FT2)/(AT2-AT0))+1
4710      *+IUP
4720 C
4730 C          ENUMERATE POSSIBILITIES FOR CURRENT REGIONS
4740 C
4750       DO 11 K=IL,IU
4760       X1=K-1
4770       K2=X1
4780       X2=N-X1
4790       X3=NDOT1-X1
4800       X4=N-X3
4810       II=X1+1
4820       JJ=X3+1
4830       PROB=A(II,JJ)
4840       IF(PROB)15,20,15
4850 15    UP1=K2.GE.KU1
4860       PROB5=PROB*.25
4870       DOWN1=K2.LE.KL1
4880       UP2=K2.GE.KU2
4890       DOWN2=K2.LE.KL2
4900 C
4910 C          DETERMINE PROPER ACTION.
4920 C
4930       IF(DOWN1.AND.DOWN2) GO TO 200
4940       IF(UP1.AND.UP2) GO TO 400
4950       IF(UP1.AND.DOWN2) GO TO 300
4960 C
4970 C          IF THIS IS A CONTINUATION POINT, SEND PROBABILITIES TO NEXT ST
4980 C
4990       B(II+1,JJ)=B(II+1,JJ)+PROB5
5000       B(II,JJ+1)=B(II,JJ+1)+PROB5
5010       B(II,JJ)=B(II,JJ)+PROB5
5020       B(II+1,JJ+1)=B(II+1,JJ+1)+PROB5
```

```
5030        GO TO 20
5040 C
5050 C          ACCUMULATE PROBABILITIES FOR A TERMINATION POINT.
5060 C
5070 200    DO 1099 IV=1,NALT
5080        ACC1(IV)=ACC1(IV)+PROB
5090       @*COEF(PA(IV),PB(IV),PAB(IV),PN(IV),Q)
5100 1099   CONTINUE
5110        GO TO 20
5120 300    DO 1199 IV=1,NALT
5130        ACC0(IV)=ACC0(IV)+PROB
5140       @*COEF(PA(IV),PB(IV),PAB(IV),PN(IV),Q)
5150 1199   CONTINUE
5160        GO TO 20
5170 400    DO 1299 IV=1,NALT
5180        ACC2(IV)=ACC2(IV)+PROB
5190       @*COEF(PA(IV),PB(IV),PAB(IV),PN(IV),Q)
5200 1299   CONTINUE
5210 20     CONTINUE
5220 11     CONTINUE
5230 22     CONTINUE
5240 C
5250 C          ACCUMULATE PROBABILITIES AND EXPECTED VALUES.
5260 C
5270        DO 2590 IV=1,NALT
5280        T9=ACC0(IV)+ACC1(IV)+ACC2(IV)
5290        ACC1T(IV)=ACC1T(IV)+ACC1(IV)
5300        ACC0T(IV)=ACC0T(IV)+ACC0(IV)
5310        ACC2T(IV)=ACC2T(IV)+ACC2(IV)
5320        NT9=N*T9
5330        N9(IV)=N9(IV)+NT9
5340 2590   CONTINUE
5350        IF(N.EQ.M0) GO TO 445
5360 C
5370 C          MOVE PROBABILITIES BACK FOR THE NEXT STEP.
5380 C
5390        DO 44 I=1,I3
5400        DO 44 J=1,I3
5410 44     A(I,J)=B(I,J)
5420 445    CONTINUE
5430 34     CONTINUE
5440        DO 6258 I=1,NALT
5450        P1=EXP(PA(I))
5460        P2=EXP(PAB(I))
5470        T=P1*(1.-P2)/(P2*(1.-P1))
5480        IF(IPUN.EQ.1)WRITE(IPNOUT,126)P1,P2,T,ACC1T(I),ACC0T(I),ACC2T(I),N
5490       @9(I),PCH(I)
5500                     WRITE(I   OUT,126)P1,P2,T,ACC1T(I),ACC0T(I),ACC2T(I),N
5510       @9(I),PCH(I)
5520 126    FORMAT(1X,2F6.2,F10.5,3F10.5,F10.2,F10.5)
5530 6258   CONTINUE
```

```
5540       STOP
5550       END
5560 C*************************************************************************
5570 C*
5580 C*     THIS PROGRAM FIGURES AND EVALUATES REGIONS FOR A SEQUENTIAL TEST
5590 C*    OF A 2X2 CONTINGENCY TABLE.  THE TEST IS BASED ON THE CROSS
5600 C*    PRODUCT RATIO.  TRUNCATION OF REGIONS IS ALLOWED. THE MARGINAL
5610 C*    PROBABILITIES ARE ASSUMED TO BE UNKNOWN. THIS PROGRAM IS FOR A
5620 C*    THREE DECISION TEST.
5630 C*
5640 C*                  WILLIAM Q. MEEKER, JR.
5650 C*                  INSTITUTE OF ADMINISTRATION AND MANAGEMENT
5660 C*                  UNION COLLEGE
5670 C*                  SCHENECTADY, NEW YORK 12308
5680 C*                  AUGUST 1974
5690 C*
5700 C*
5710 C*************************************************************************
5720 C
5730       DIMENSION A(27,27),B(27,27,2)
5740       DIMENSION P1(100),P2(100),P3(100),P4(100)
5750       DIMENSION ACC1(100),ACC2(100),ACC0(100)
5760       DIMENSION ACC1T(100),ACC2T(100),ACC0T(100),PCH(100),N9(100)
5770       REAL N9,NT9
5780       LOGICAL UP1,UP2,DOWN1,DOWN2
5790       COMMON IL,IU,G1,G2,G3,G4,X1,X2,X3,X4
5800       DATA T0/1./
5810       DATA AT0,I1,I2/0.,1,2/
5820       DATA ITOUT,ITIN,IPOUT,INPUT,IOUT/90,91,58,50,66/
5830 C
5840 C         SPECIFY H1(LOWER) AND H2(UPPER).  H0 IS ASSUMED TO BE T=1.
5850 C         TAKE LOGS FOR LATER USE
5860 C
5870       READ(INPUT,1212)IPUN,IREC
5880 1212  FORMAT(2I1)
5890       IUP=0
5900       READ(INPUT,70)ALPHA1,BETA1,ALPHA2,BETA2,XM0,T2,T0,T1
5910       T1=1./T2
5920       T0=1.
5930       M0=XM0
5940       AT1=ALOG(T1)
5950       AT2=ALOG(T2)
5960 C
5970 C         SET DESIRED ERROR PROBABILITIES, CRITICAL LIMITS AND THEIR LOG
5980 C
5990       A1=ALPHA1/(1.0-BETA1)
6000       A2=BETA2/(1.0-ALPHA2)
6010       B1=(1.0-ALPHA1)/BETA1
6020       B2=(1.0-BETA2)/ALPHA2
6030       WRITE(IOUT,47)
6040 47    FORMAT(1H1,6X,6HNUMBER,5X,3HLOG/)
```

```
      WRITE(IOUT,41)T0,AT0,T1,AT1,T2,AT2
41    FORMAT(3H T0,2X,2(3X,F7.4)/3H T1,2X,2(3X,F7.4)/3H T2,2X,2(3X,F7.4)
     &)
      WRITE(IOUT,45)ALPHA1,BETA1,ALPHA2,BETA2
45    FORMAT(///10H ALPHA1 = ,F5.3/9H BETA1 = ,F5.3
     &/10H ALPHA2 = ,F5.3/9H BETA2 = ,F5.3 ///)
C
C          READ SELECTED ALTERNATE HYPOTHESES WHERE REGIONS ARE TO BE EVA
C
      I=0
1     CONTINUE
      I=I+1
      READ(INPUT,70)P1(I),P2(I),P3(I),P4(I)
      IF(P1(I).EQ.0.0) GO TO 9995
      P4(I)=1.-P1(I)-P2(I)-P3(I)
      GO TO 1
9995  NALT=I-1
      DO 2265 I=1,NALT
      P1(I)=ALOG(P1(I))
      P2(I)=ALOG(P2(I))
      P3(I)=ALOG(P3(I))
      P4(I)=ALOG(P4(I))
2265  CONTINUE
      AL1=ALOG(A1)
      AL2=ALOG(A2)
      BL1=ALOG(B1)
      BL2=ALOG(B2)
70    FORMAT(8F10.0)
        CALL SETSCT(ITOUT,1)
        CALL SETSCT(ITIN,1)
C
C          WRITE PROBABILITIES FOR THE FIRST STEP
C
      B(1,1,1)=.25
      B(1,2,1)=.25
      B(1,1,2)=.25
      B(2,2,2)=.25
      WRITE(ITIN    )((B(KK,JK,1),KK=1,2),JK=1,2)
      WRITE(ITIN    )((B(KK,JK,2),KK=1,2),JK=1,2)
         CALL SETSCT(ITOUT,1)
        CALL SETSCT(ITIN,1)
      DO 900 I=1,NALT
      N9(I)=0.0
      ACC1T(I)=0.0
      ACC0T(I)=0.0
      ACC2T(I)=0.0
900   CONTINUE
C
C          <<<  INCREMENT TRIAL NUMBER  >>>
C
      DO 34 N=1,M0
```

```
      WRITE(IOUT,5637)N
5637  FORMAT("   NOW AT TRIAL  ",I5)
      Q=FLOAT(N)*(-1.386294)
      N1=N+1
      I3=N+2
      DO 522 I=1,NALT
      ACC1(I)=0.0
      ACC2(I)=0.0
      ACC0(I)=0.0
522   CONTINUE
      IF(N.NE.MO) GO TO 56
      IF(IREC.NE.0)PRINT 1777
      IF(IREC.NE.0)IUP=1
1777  FORMAT(" REGION MOVE")
C
C         ALLOW TRUNCATION IF DESIRED
C
      AL1=(AL1+BL1)/2.0
      AL2=(AL2+BL2)/2.0
      BL1=AL1
      BL2=AL2
      DO 2777 I=1,NALT
2777  PCH(I)=1.-ACC1T(I)-ACC2T(I)-ACC0T(I)
56    CONTINUE
      DO 78 I=1,I3
      DO 78 J=1,I3
78    B(I,J,I1)=0.0
C
C         <<<  ENUMERATE ALL POSSIBLE MARGINS  >>>
C
      DO 33 I=1,N1
      DO 77 K=1,I3
      DO 77 J=1,I3
77    B(K,J,I2)=0.0
      READ(ITIN)((A(KK,JK),KK=1,N1),JK=1,N1)
      N1DOT=I-1
      DO 22 J=1,N1
      NDOT1=J-1
C
C         FIGURE LOWER AND UPPER LIMITS ON N11
C
      IU=MINO(NDOT1,N1DOT)+1
      IL=MAXO(NDOT1+N1DOT-N,0)+1
      FT1=FNOD(NDOT1,N1DOT,N,AT1)
      FT2=FNOD(NDOT1,N1DOT,N,AT2)
C
C         FIGURE CRITICAL VALUES OF REGIONS
C
      KL1=ILNT(-(AL1+FT1)/AT1)-IUP
      KU1=ILNT(-(BL1+FT1)/AT1)-IUP+1
      KU2=ILNT((BL2-FT2)/AT2)+IUP+1
```

```
      KL2=ILNT((AL2-FT2)/AT2)+1UP
C
C          <<<  ENUMERATE POSSIBILITIES FOR CURRENT REGIONS  >>>
C
      DO 11 K=IL,IU
      K2=K-1
        X1=K2
      X2=N1DOT-K2
      X3=NDOT1-K2
      X4=N-N1DOT-NDOT1+K2
      PROB=A(K,J)
      IF(PROB)15,20,15
15    UP1=K2.GE.KU1
      PROB25=PROB*.25
      DOWN2=K2.LE.KL2
C
C          DETERMINE PROPER ACTION.
C
      IF((UP1.AND.DOWN2)) GO TO 400
      DOWN1=K2.LE.KL1
      IF(DOWN1.AND.DOWN2) GO TO 200
      UP2=K2.GE.KU2
      IF(UP1.AND.UP2) GO TO 300
C
C          IF THIS IS A CONTINUATION POINT, SEND PROBABILITIES TO NEXT STE
C
      B(K,J,I2)=B(K,J,I2)+PROB25
      B(K,J,I1)=B(K,J,I1)+PROB25
      B(K,J+1,I1)=B(K,J+1,I1)+PROB25
      B(K+1,J+1,I2)=B(K+1,J+1,I2)+PROB25
      GO TO 20
C
C          ACCUMULATE PROBABILITIES FOR A TERMINATION POINT.
C
200   DO 8001  IV=1,NALT
      ACC1(IV)=ACC1(IV)+PROB*COEF(P1(IV),P2(IV),P3(IV),P4(IV),Q)
8001  CONTINUE
      GO TO 20
300   DO 8002  IV=1,NALT
      ACC2(IV)=ACC2(IV)+PROB*COEF(P1(IV),P2(IV),P3(IV),P4(IV),Q)
8002  CONTINUE
      GO TO 20
400   DO 8003 IV=1,NALT
      ACC0(IV)=ACC0(IV)+PROB*COEF(P1(IV),P2(IV),P3(IV),P4(IV),Q)
8003  CONTINUE
20    CONTINUE
11    CONTINUE
22    CONTINUE
      WRITE(ITOUT    )((B(KJ,JK,I1),KJ=1,I3),JK=1,I3)
      IHOLD=I1
      I1=I2
```

```
7580        I2=IHOLD
7590 33     CONTINUE
7600        WRITE(ITOUT   )((B(KK,JK,I1),KK=1,I3),JK=1,I3)
7610           CALL SETSCT(ITOUT,1)
7620         CALL SETSCT(ITIN,1)
7630        IHOLDF=ITIN
7640        ITIN=ITOUT
7650        ITOUT=IHOLDF
7660 C
7670 C          ACCUMULATE PROBABILITIES AND EXPECTED VALUES.
7680 C
7690        DO 8005 IV=1,NALT
7700        T9=ACC0(IV)+ACC1(IV)+ACC2(IV)
7710        ACC2T(IV)=ACC2T(IV)+ACC2(IV)
7720        ACC1T(IV)=ACC1T(IV)+ACC1(IV)
7730        ACC0T(IV)=ACC0T(IV)+ACC0(IV)
7740        NT9=FLOAT(N)*T9
7750        N9(IV)=N9(IV)+NT9
7760 8005   CONTINUE
7770 34     CONTINUE
7780        DO 6565 I=1,NALT
7790        P1(I)=EXP(P1(I))
7800        P2(I)=EXP(P2(I))
7810        P3(I)=EXP(P3(I))
7820 6565   P4(I)=EXP(P4(I))
7830         DO 3459 I=1,NALT
7840         WRITE(ICUT,4562)P1(I),P2(I),P3(I),P4(I),ACC1T(I)
7850       &     ,ACC0T(I),ACC2T(I),N9(I)
7860       @,PCH(I)
7870         IF(IPUN.EQ.1)
7880       &  WRITE(IPOUT,4562)P1(I),P2(I),P3(I),P4(I),ACC1T(I)
7890       &     ,ACC0T(I),ACC2T(I),N9(I)
7900       @,PCH(I)
7910 4562   FORMAT(1X,4F6.3,3F10.5,F12.3,F10.5)
7920 3459   CONTINUE
7930 99     PRINT 456
7940 456    FORMAT("0END OF RUN")
7950        STOP
7960        END
7970 C
7980 C*****************************************************************************
7990 C*
8000 C*      THIS PROGRAM FIGURES AND EVALUATEST REGIONS FOR A SEQUENTIAL TEST
8010 C*    OF A 2X2 CONTINGENCY TABLE WITH KNOWN MARGINAL PROBABILITIES.
8020 C*    THIS PROGRAM IS FOR A THREE DECISION TEST PROCEDURE.
8030 C*    TRUNCATION OF THE TEST IS ALLOWED.
8040 C*
8050 C*                    WILLIAM Q. MEEKER, JR.
8060 C*                    INSTITUTE OF ADMINISTRATION AND MANAGEMENT
8070 C*                    UNION COLLEGE
8080 C*                    SCHENECTADY, NEW YORK 12308
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```
C*                    AUGUST 1974
C*
C*****************************************************************************
C
      DIMENSION A(27,27),B(27,27,2)
      DIMENSION P1(20),P2(20),P3(20),P4(20)
      DIMENSION ACC1(20),ACC2(20),ACC0(20)
      DIMENSION ACC1T(20),ACC2T(20),ACC0T(20),N9(20)
      DIMENSION PCH(20)
      EQUIVALENCE(AT1,XM0)
      REAL N9,NT9
      LOGICAL UP1,UP2,DOWN1,DOWN2
      COMMON IL,IU,G1,G2,G3,G4,X1,X2,X3,X4
      DATA ITOUT,ITIN,IPOUT,INPUT,IOUT/90,91,58,50,66/
      DATA AT0,I1,I2/0.,1,2/
C
C         SPECIFY H1(LOWER) AND H2(UPPER).  H0 IS ASSUMED TO BE T=1.
C         TAKE LOGS FOR LATER USE
C
      READ(INPUT,1212)IPUN,IREC
1212  FORMAT(2I1)
      IUP=0
      READ(INPUT,70)ALPHA1,BETA1,ALPHA2,BETA2,XM0
106   READ(INPUT,70)P1DOT,PDOT1,QH1,PH1,SH1
      IF(SH1.EQ.0.0)SH1=P1DOT*PDOT1
      M0=XM0
      PH2=P1DOT-PH1
      PH3=PDOT1-PH1
      PH4=1.-PDOT1-P1DOT+PH1
      QH2=P1DOT-QH1
      QH4=1.-P1DOT-PDOT1+QH1
      QH3=PDOT1-QH1
      SH2=P1DOT-SH1
      SH3=PDOT1-SH1
      SH4=1.-P1DOT-PDOT1+SH1
      XU1=ALOG(PH1/SH1)
      XU2=ALOG(PH2/SH2)
      XU3=ALOG(PH3/SH3)
      XU4=ALOG(PH4/SH4)
      ZU=XU1-XU2-XU3+XU4
      XL1=ALOG(SH1/QH1)
      XL2=ALOG(SH2/QH2)
      XL3=ALOG(SH3/QH3)
      XL4=ALOG(SH4/QH4)
      ZL=XL1-XL2-XL3+XL4
      XL43=XL4-XL3
      XL42=XL4-XL2
      XU43=XU4-XU3
      XU42=XU4-XU2
      WRITE(IOUT,47)QH1,QH2,QH3,QH4,SH1,SH2,SH3,SH4,PH1,PH2,PH3,PH4
```

```
 47    FORMAT("0       P11     P12     P21     P22"/
     &" H1 ",4F8.3/" H0 ",4F8.3/" H2 ",4F8.3)
C
C          SET DESIRED ERROR PROBABILITIES, CRITICAL LIMITS AND THEIR LOG
C
       A1=ALPHA1/(1.0-BETA1)
       A2=BETA2/(1.0-ALPHA2)
       B1=(1.0-ALPHA1)/BETA1
       B2=(1.0-BETA2)/ALPHA2
       WRITE(IOUT,45)ALPHA1,BETA1,ALPHA2,BETA2
 45    FORMAT(///10H ALPHA1 = ,F5.3/9H BETA1 = ,F5.3
     &/10H ALPHA2 = ,F5.3/9H BETA2 = ,F5.3 ///)
       I=0
C
C          READ SELECTED ALTERNATE HYPOTHESES WHERE REGIONS ARE TO BE EVA
C
 1     CONTINUE
       I=I+1
       READ(INPUT,70)P1(I)
       IF(P1(I).EQ.0.0) GO TO 9995
       P2(I)=P1DOT-P1(I)
       P3(I)=PDOT1-P1(I)
       P4(I)=1.0-PDOT1-P1DOT+P1(I)
       GO TO 1
 9995  NALT=I-1
       DO 2265 I=1,NALT
       P1(I)=ALOG(P1(I))
       P2(I)=ALOG(P2(I))
       P3(I)=ALOG(P3(I))
       P4(I)=ALOG(P4(I))
 2265  CONTINUE
       AL1=ALOG(A1)
       AL2=ALOG(A2)
       BL1=ALOG(B1)
       BL2=ALOG(B2)
 70    FORMAT(8F10.0)
          CALL SETSCT(ITOUT,1)
       CALL SETSCT(ITIN,1)
C
C          WRITE PROBABILITIES FOR THE FIRST STEP
C
       B(2,2,2)=.25
       B(1,1,2)=.25
       B(1,2,1)=.25
       B(1,1,1)=.25
       WRITE(ITIN   )((B(KK,JK,1),KK=1,2),JK=1,2)
       WRITE(ITIN   )((B(KK,JK,2),KK=1,2),JK=1,2)
          CALL SETSCT(ITOUT,1)
       CALL SETSCT(ITIN,1)
       DO 900 I=1,NALT
       N9(I)=0.0
```

```
9110          ACC1T(I)=0.0
9120          ACC0T(I)=0.0
9130          ACC2T(I)=0.0
9140  900     CONTINUE
9150  C
9160  C           <<<  INCREMENT TRIAL NUMBER  >>>
9170  C
9180          DO 34 N=1,M0
9190          WRITE(IOUT,5637)N
9200  5637    FORMAT("   NOW AT TRIAL  ",I5)
9210          Q=FLOAT(N)*(-1.386294)
9220          XN=N
9230          N1=N+1
9240          I3=N+2
9250          DO 522 I=1,NALT
9260          ACC1(I)=0.0
9270          ACC2(I)=0.0
9280          ACC0(I)=0.0
9290  522     CONTINUE
9300          IF(N.NE.M0) GO TO 56
9310  C
9320  C           ALLOW TRUNCATION IF DESIRED
9330  C
9340          DO 2777 I=1,NALT
9350  2777    P(I)=1.-ACC1T(I)-ACC2T(I)-ACC0T(I)
9360          AL1=(AL1+BL1)/2.0
9370          AL2=(AL2+BL2)/2.0
9380          BL1=AL1
9390          BL2=AL2
9400          IF(IREC.NE.0)PRINT 1777
9410          IF(IREC.NE.0)IUP=1
9420  1777    FORMAT(" REGION MOVE")
9430  56      CONTINUE
9440          DO 78 I=1,I3
9450          DO 78 J=1,I3
9460  78      B(I,J,I1)=0.0
9470  C
9480  C           <<<  ENUMERATE ALL POSSIBLE MARGINS  >>>
9490  C
9500          DO 33 I=1,N1
9510          DO 77 K=1,I3
9520          DO 77 J=1,I3
9530  77      B(K,J,I2)=0.0
9540          READ(ITIN)((A(KK,JK),KK=1,N1),JK=1,N1)
9550          N1DOT=I-1
9560          XN1DOT=N1DOT
9570          DO 22 J=1,N1
9580          NDOT1=J-1
9590          XNDOT1=NDOT1
9600          XNL=XN1DOT*XL42+XNDOT1*XL43-XN*XL4
9610          XNU=XN1DOT*XU42+XNDOT1*XU43-XN*XU4
```

```
9620 C
9630 C          FIGURE LOWER AND UPPER LIMITS ON N11
9640 C
9650         IU=MINO(NDOT1,N1DOT)+1
9660         IL=MAXO(NDOT1+N1DOT-N,0)+1
9670 C
9680 C          SKIP IF REGIONS ARE KNOWN AND ALREADY SAVED ON UNIT ITREG
9690 C
9700 C
9710 C          FIGURE CRITICAL VALUES OF REGIONS
9720 C
9730         KL1=ILNT((AL1+XNL)/ZL)-IUP
9740         KU1=ILNT((BL1+XNL)/ZL)-IUP+1
9750         KL2=ILNT((AL2+XNU)/ZU)+IUP
9760         KU2=ILNT((BL2+XNU)/ZU)+IUP+1
9770 C
9780 C          <<<  ENUMERATE POSSIBILITIES FOR CURRENT REGIONS  >>>
9790 C
9800         DO 11 K=IL,IU
9810         X1=K-1
9820         K2=X1
9830         X2=N1DOT-K2
9840         X3=NDOT1-K2
9850         X4=N-N1DOT-NDOT1+K2
9860         PROB=A(K,J)
9870         IF(PROB)15,20,15
9880 15      UP1=K2.GE.KU1
9890         PROB25=PROB*.25
9900         DOWN2=K2.LE.KL2
9910 C
9920 C          DETERMINE PROPER ACTION.
9930 C
9940         IF((UP1.AND.DOWN2)) GO TO 400
9950         DOWN1=K2.LE.KL1
9960         IF(DOWN1.AND.DOWN2) GO TO 200
9970         UP2=K2.GE.KU2
9980         IF(UP1.AND.UP2) GO TO 300
9990 C
10000 C      IF THIS IS A CONTINUATION POINT, SEND PROBABILITIES TO NEXT STEP
10010 C
10020        B(K,J,I2)=B(K,J,I2)+PROB25
10030        B(K+1,J+1,I2)=B(K+1,J+1,I2)+PROB25
10040        B(K,J+1,I1)=B(K,J+1,I1)+PROB25
10050        B(K,J,I1)=B(K,J,I1)+PROB25
10060        GO TO 20
10070 C
10080 C          ACCUMULATE PROBABILITIES FOR A TERMINATION POINT.
10090 C
10100 200    DO 8001 IV=1,NALT
10110         ACC1(IV)=ACC1(IV)+PROB*COEF(P1(IV),P2(IV),P3(IV),P4(IV),0)
10120 8001   CONTINUE
```

```
10130          GO TO 20
10140 300      DO 8002 IV=1,NALT
10150          ACC2(IV)=ACC2(IV)+PROB*COEF(P1(IV),P2(IV),P3(IV),P4(IV),Q)
10160 8002     CONTINUE
10170          GO TO 20
10180 400      DO 8003 IV=1,NALT
10190          ACC0(IV)=ACC0(IV)+PROB*COEF(P1(IV),P2(IV),P3(IV),P4(IV),Q)
10200 8003     CONTINUE
10210 20       CONTINUE
10220 11       CONTINUE
10230 22       CONTINUE
10240          WRITE(ITOUT    )((B(KJ,JK,I1),KJ=1,I3),JK=1,I3)
10250          IHOLD=I1
10260          I1=I2
10270          I2=IHOLD
10280 33       CONTINUE
10290          WRITE(ITOUT    )((B(KK,JK,I1),KK=1,I3),JK=1,I3)
10300             CALL SETSCT(ITOUT,1)
10310          CALL SETSCT(ITIN,1)
10320          IHOLDF=ITIN
10330          ITIN=ITOUT
10340          ITOUT=IHOLDF
10350 C
10360 C            ACCUMULATE PROBABILITIES AND EXPECTED VALUES.
10370 C
10380          DO 8005 IV=1,NALT
10390          T9=ACC0(IV)+ACC1(IV)+ACC2(IV)
10400          ACC2T(IV)=ACC2T(IV)+ACC2(IV)
10410          ACC1T(IV)=ACC1T(IV)+ACC1(IV)
10420          ACC0T(IV)=ACC0T(IV)+ACC0(IV)
10430          NT9=FLOAT(N)*T9
10440          N9(IV)=N9(IV)+NT9
10450 8005     CONTINUE
10460 34       CONTINUE
10470          DO 6565 I=1,NALT
10480          P1(I)=EXP(P1(I))
10490          P2(I)=EXP(P2(I))
10500          P3(I)=EXP(P3(I))
10510          P4(I)=EXP(P4(I))
10520 6565     CONTINUE
10530          DO 3459 I=1,NALT
10540          WRITE(IOUT,4562)P1(I),P2(I),P3(I),P4(I),ACC1T(I)
10550         &    ,ACC0T(I),ACC2T(I),N9(I)
10560         @,PCH(I)
10570          IF(IPUN.EQ.1)
10580         & WRITE(IPOUT,4562)P1(I),P2(I),P3(I),P4(I),ACC1T(I)
10590         &    ,ACC0T(I),ACC2T(I),N9(I)
10600         @,PCH(I)
10610 4562     FORMAT(1X,4F6.3,3F10.5,F12.3,F10.5)
10620 3459     CONTINUE
10630 99       PRINT 456
```

```
10640 456    FORMAT("0END OF RUN")
10650        STOP
10660        END
10670 C
10680 C*************************************************************
10690 C*
10700 C*    SUBROUTINES USED IN THE PROGRAMS FOR SEQUENTIAL ANALYSIS
10710 C*    OF 2x2 CONTINGENCY TABLES.
10720 C*
10730 C*************************************************************
10740 C
10750 C
10760 C      RETURN LOG BINOMIAL COEFFICIENT
10770 C
10780        FUNCTION BICOF(N,IR)
10790        BICOF=FLNG(N)-FLNG(IR)-FLNG(N-IR)
10800        RETURN
10810        END
10820 C
10830 C      FUNCTION TO DETERMINE THE LIKLIHOOD RATIO
10840 C
10850        FUNCTION FNOD(NDOT1,N1DOT,N,AT)
10860        COMMON IL,IU
10870        TOPNUM=BICOF(N,NDOT1)
10880        FNOD=0.
10890        DO 22 I=IL,IU
10900        J=I-1
10910        FNOD=FNOD+EXP(BICOF(N1DOT,J)+BICOF(N-N1DOT,NDOT1-J)+FLOAT(J)*AT-
10920       &    TOPNUM)
10930 22     CONTINUE
10940        FNOD=-ALOG(FNOD)
10950        RETURN
10960        END
10970         FUNCTION COEF(P1,P2,P3,P4,Q)
10980         COMMON IL,IU,G(4),X1,X2,X3,X4
10990         COFF=EXP(X1*P1+X2*P2+X3*P3+X4*P4-Q)
11000         RETURN
11010         END
11020 C
11030 C          FUNCTION TO RETURN THE NATURAL LOG FACTORIAL
11040 C
11050 C
11060 C      RETURN LOG BINOMIAL COEFFICIENT
11070 C
11080        FUNCTION BICOF(N,IR)
11090        BICOF=FLNG(N)-FLNG(IR)-FLNG(N-IR)
11100        RETURN
11110        END
11120 C
11130 C      FUNCTION TO DETERMINE THE LIKLIHOOD RATIO
11140 C
```

```
11150 C
11160 C
11170 C          FUNCTION TO RETURN THE NATURAL LOG FACTORIAL
11180 C
11190         FUNCTION FLNG(J)
11200         DIMENSION F(105)
11210         DATA MARK/1/
11220         IF(MARK)20,20,21
11230 20      FLNG=F(J+1)
11240         RETURN
11250 21      F(1)=0.
11260         F(2)=0.
11270         DO 22 I=3,103
11280         F(I)=F(I-1)+ALOG(FLOAT(I-1))
11290 22      CONTINUE
11300         MARK=0
11310         GO TO 20
11320         END
11330 C
11340 C       THIS FUNCTION RETURNS THE GREATEST INTEGER .LE. X
11350         FUNCTION ILNT(X)
11360         X=X+.001
11370         IF(X)1,2,3
11380 1       ILNT=IFIX(X)-1
11390         GO TO 2
11400 3       ILNT=IFIX(X)
11410 2       RETURN
11420         END
```